ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$ctg^2A+ctg^2B+ctg^2C\le 9\left(\frac{R}{2r}\right)^4-8$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}ctg^2A=\sum_{cyc}\left(\frac{b^2+c^2-a^2}{4F}\right)^2=\frac{1}{16F^2}\sum_{cyc}(b^2+c^2-a^2)^2=$$

$$\frac{1}{16F^2}[(b^4+c^4+a^4+2b^2c^2-2a^2c^2-2a^2b^2)+$$

$$+(b^4+c^4+a^4+2a^2b^2-2a^2c^2-2b^2c^2)+$$

$$+(b^4+c^4+a^4+2a^2c^2-2b^2c^2-2a^2b^2)]=$$

$$\frac{1}{16F^2}[3(b^4+c^4+a^4)-2(a^2b^2+b^2c^2+a^2c^2)]=$$

$$\frac{1}{16F^2}(3(b^4+c^4+a^4+2a^2b^2+2b^2c^2+2a^2c^2)-$$

$$-8(a^2b^2+b^2c^2+a^2c^2))=$$

$$\frac{1}{16F^2}\left(3(b^2+c^2+a^2)^2-8(a^2b^2+b^2c^2+a^2c^2)\right)$$

$$\boxed{b^2+c^2+a^2\le 9R^2}\quad (Leibniz)$$

$$\boxed{a^2b^2+b^2c^2+a^2c^2\ge 16F^2}\quad (Goldner\ \ 2)$$

$$\sum_{cyc}ctg^2A\le\frac{1}{16F^2}(3.(9R^2)^2-8.16F^2)=\frac{243R^4}{16F^2}-8$$

$$\le\frac{243R^4}{16.s^2.r^2}-8\overset{s\le 3\sqrt{3}r}{\leq}\frac{243R^4}{16.27.r^2.r^2}-8=9\left(\frac{R}{2r}\right)^4-8$$

$$ctg^2A+ctg^2B+ctg^2C\le 9\left(\frac{R}{2r}\right)^4-8$$