ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc} sin^3(A)\cos(B)\cos(C) \le \frac{9\sqrt{3}}{32}$$

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$$\sum_{cyc} sin^3(A).\cos(B).\cos(C) = \sum_{cyc} sin^2(A).(\sin(A).\cos(B).\cos(C)) =$$

$$\sum_{cyc}\left(1 - cos^2(A)\right)(\sin(A).\cos(B).\cos(C)) =$$

$$\sum_{cyc}(\sin(A).\cos(B).\cos(C)) - \sum_{cyc} cos^2(A).(\sin(A).\cos(B).\cos(C))$$

$$(\sin(A).\cos(B).\cos(C)) = \frac{1}{2}\cos(C).(\sin(A+B) + \sin(A-B)) =$$

$$\frac{1}{2}\cos(C).(\sin(C) + \sin(A-B)) = \frac{1}{2}(\cos(C).\sin(C) + \cos(C).\sin(A-B)) =$$

$$= \frac{1}{2}(\frac{1}{2}\sin(2C) + \frac{1}{2}(\sin(A-B+C) + \sin(A-B-C)) = \frac{1}{4}(\sin(2C) + \sin(\pi - 2B) - \sin(2A)) =$$

$$= \frac{1}{4}(\sin(2C) + \sin(2B) - \sin(2A))$$

$$\Omega_1 = \sum_{cyc}(\sin(A).\cos(B).\cos(C)) = \sum_{cyc}\frac{1}{4}(\sin(2C) + \sin(2B) - \sin(2A)) =$$

$$= \frac{1}{4}(\sin(2C) + \sin(2B) + \sin(2A))$$

$$\Omega_2 = \sum_{cyc} cos^2(A).(\sin(A).\cos(B).\cos(C)) =$$

$$\sum_{cyc} cos^2(A).\frac{1}{4}(\sin(2C) + \sin(2B) - \sin(2A)) = \frac{1}{4}.\frac{1}{2}\sum_{cyc}(1 + \cos(2A))(\sin(2C) + \sin(2B) - \sin(2A)) =$$

$$= \frac{1}{8}\sum_{cyc}(\sin(2C) + \sin(2B) - \sin(2A) + \sin(2C).\cos(2A) +$$

$$\sin(2B).\cos(2A) - \sin(2A).\cos(2A)) =$$

$$=\frac{1}{8}((\sin(2C)+\sin(2B)-\sin(2A))+(\sin(2A)+\sin(2C)-\sin(2B))+$$

$$(\sin(2A)+\sin(2B)-\sin(2C))+(\sin(2C).\cos(2A)+\sin(2A).\cos(2C))+$$

$$(\sin(2B).\cos(2A)+\sin(2A).\cos(2B))+(\sin(2B).\cos(2C)+\sin(2C).\cos(2B)-$$

$$(\sin(2A).\cos(2A)+\sin(2B).\cos(2B)+\sin(2C).\cos(2C)))=$$

$$\frac{1}{8}((\sin(2A)+\sin(2B)+\sin(2C))+\sin(2A+2C)+\sin(2A+2B)+$$

$$\sin(2B+2C)-\frac{1}{2}((\sin(4A)+\sin(4B)+\sin(4C)))=$$

$$\frac{1}{8}((\sin(2A)+\sin(2B)+\sin(2C))+\sin(2\pi-2B)+\sin(2\pi-2A)+$$

$$\sin(2\pi-2C)-\frac{1}{2}((\sin(4A)+\sin(4B)+\sin(4C)))=$$

$$\frac{1}{8}((\sin(2A)+\sin(2B)+\sin(2C))-\sin(2B)-\sin(2A)-\sin(2C)-$$

$$\frac{1}{2}((\sin(4A)+\sin(4B)+\sin(4C)))=-\frac{1}{16}((\sin(4A)+\sin(4B)+\sin(4C)))$$

$$\sum_{cyc} sin^3(A).\cos(B).\cos(C)=\frac{1}{4}\sum_{cyc}\sin(2A)+\frac{1}{16}\sum_{cyc}\sin(4A)$$

For the functions $f(x)=\sin(2x)$ *and* $f(x)=\sin(4x)$

$x\in(0;\pi)$, $f''(x)=-4\sin(2x)<0$; $f''(x)=-16\sin(4x)<0$

That's why $f(x)=\sin(2x)$, $f(x)=\sin(4x)$ *is concave*

functions. Then, according to Jensen's theorem

$$\sum_{cyc}\sin(2A)\leq\frac{3\sqrt{3}}{2}\ \ and\ \ \sum_{cyc}\sin(4A)\leq-\frac{3\sqrt{3}}{2}$$

$$\sum_{cyc} sin^3(A).\cos(B).\cos(C)\leq\frac{1}{4}.\frac{3\sqrt{3}}{2}-\frac{1}{16}.\frac{3\sqrt{3}}{2}=\frac{3\sqrt{3}}{8}-\frac{3\sqrt{3}}{32}=\frac{9\sqrt{3}}{32}$$

Equality holds for $A=B=C$.