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In ΔABC the following relationship holds:

$$\frac{r_a^2 r_b r_c}{r_b+r_c+r_b r_c h_a}+\frac{r_b^2 r_a r_c}{r_a+r_c+r_a r_c h_b}+\frac{r_c^2 r_b r_a}{r_b+r_a+r_b r_a h_c}\geq \frac{54r^2 S}{2\sqrt{3}R+\frac{abc\left(w_a^3+w_b^3+w_c^3\right)}{4w_a w_b w_c}}$$

Proposed by Elsen Kerimov-Azerbaijan

Solution by Avishek Mitra-India

$$\sum_{cyc}\frac{r_a^2 r_b r_c}{r_b+r_c+r_b r_c h_a}=\sum_{cyc}\frac{r_a^2}{\frac{1}{r_c}+\frac{1}{r_b}+h_a}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc} r_a\right)^2}{2\sum_{cyc}\frac{1}{r_a}+\sum_{cyc} h_a}=$$

$$=\frac{(4R+r)^2}{\frac{2}{r}+\sum_{cyc} h_a}\overset{Euler}{\geq}\frac{81r^2}{\frac{2}{r}+\sum_{cyc} h_a}$$

$$\frac{54r^2\Delta}{2\sqrt{3}R+\frac{abc\sum_{cyc} w_a^3}{4\prod w_a}}\overset{AM-GM}{\leq}\frac{54r^2\Delta}{2\sqrt{3}R+\frac{3abc}{4}}\overset{Mitrinovic}{\leq}\frac{54r^3 s}{2\sqrt{3}.\frac{2s}{3\sqrt{3}}+\frac{3.4Rrs}{4}}=$$

$$\frac{54r^3}{\frac{4}{3}+3Rr}\to \ Need\ to\ show \to \frac{81r^2}{\frac{2}{r}+\sum_{cyc} h_a}\geq \frac{54r^3}{\frac{4}{3}+3Rr}\to$$

$$4+9Rr\geq 4+2r\sum_{cyc} h_a \to \sum_{cyc} h_a \leq \frac{9R}{2}\quad (*\,True)$$

$$\therefore \sum_{cyc} h_a \leq \sum_{cyc} m_a \overset{CBS}{\leq}\sqrt{\frac{3.3}{4}\sum_{cyc} a^2}\overset{Leibniz}{\leq}\sqrt{\frac{9}{4}.9R^2}=\frac{9R}{2}$$