ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$2\sqrt{3}s-2R-r_a+r\le\sqrt{3}(b+c)\le 8R-r_a-r$$

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$$2\sqrt{3}\mathrm{s}-2\mathrm{R}-\mathrm{r}_a+\mathrm{r}\le\sqrt{3}(\mathrm{b}+\mathrm{c})\Leftrightarrow\sqrt{3}a\le 2\mathrm{R}+\frac{\mathrm{rs}}{\mathrm{s}-a}-\frac{\mathrm{rs}}{\mathrm{s}}=2\mathrm{R}+\frac{\mathrm{r}a}{\mathrm{s}-a}$$

$$\Leftrightarrow\sqrt{3}.4\mathrm{R}\sin\frac{A}{2}\cos\frac{A}{2}\le 2\mathrm{R}+\frac{4\mathrm{R}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4\mathrm{R}\sin\frac{A}{2}\cos\frac{A}{2}}{4\mathrm{R}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow 2\sqrt{3}\sin\frac{A}{2}\cos\frac{A}{2}\le 1+2\sin^2\frac{A}{2}$$

$$\Leftrightarrow 1+4\sin^4\frac{A}{2}+4\sin^2\frac{A}{2}\ge 12\sin^2\frac{A}{2}\left(1-\sin^2\frac{A}{2}\right)\Leftrightarrow 16\sin^4\frac{A}{2}-8\sin^2\frac{A}{2}+1\ge 0$$

$$\Leftrightarrow\left(4\sin^2\frac{A}{2}-1\right)^2\ge 0\to\text{true}\therefore 2\sqrt{3}\mathrm{s}-2\mathrm{R}-\mathrm{r}_a+\mathrm{r}\le\sqrt{3}(\mathrm{b}+\mathrm{c})\ and$$

$$\sqrt{3}(\mathrm{b}+\mathrm{c})\le 8\mathrm{R}-\mathrm{r}_a-\mathrm{r}\Leftrightarrow\sqrt{3}.4\mathrm{R}\cos\frac{A}{2}\cos\frac{B-C}{2}\le 8\mathrm{R}-\frac{\mathrm{r}(\mathrm{b}+\mathrm{c})}{\mathrm{s}-a}$$

$$\Leftrightarrow\sqrt{3}.4\mathrm{R}\cos\frac{A}{2}\cos\frac{B-C}{2}\le 8\mathrm{R}-\frac{4\mathrm{R}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4\mathrm{R}\cos\frac{A}{2}\cos\frac{B-C}{2}}{4\mathrm{R}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow\left(\sqrt{3}.\cos\frac{A}{2}+\sin\frac{A}{2}\right)\cos\frac{B-C}{2}\le 2\ and\ \because 0<\cos\frac{B-C}{2}\le 1$$

$$\therefore\text{ it suffices to prove : }\sqrt{3}.\cos\frac{A}{2}+\sin\frac{A}{2}\le 2\to\text{true}\because\sqrt{3}.\cos\frac{A}{2}+\sin\frac{A}{2}\overset{\text{CBS}}{\le}$$

$$\sqrt{3+1}.\sqrt{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}=2\therefore\sqrt{3}(\mathrm{b}+\mathrm{c})\le 8\mathrm{R}-\mathrm{r}_a-\mathrm{r}\ and\text{ so,}$$

$$2\sqrt{3}\mathrm{s}-2\mathrm{R}-\mathrm{r}_a+\mathrm{r}\le\sqrt{3}(\mathrm{b}+\mathrm{c})\le 8\mathrm{R}-\mathrm{r}_a-\mathrm{r},''=''\text{ iff }\hat{A}=\frac{\pi}{3}\text{ for }1^{\text{st}}\text{ inequality}$$

$$and\text{ iff }\Delta\text{ ABC is equilateral for }2^{\text{nd}}\text{ inequality (QED)}$$