**In any $\Delta$ ABC the following relationship holds :**

$$3\sum_{cyc}\sqrt[3]{2\left(r_a^2+r_b^2\right)^2.r_c^2}\ge\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}(m_a+w_a)\right)$$

*Proposed by Dang Ngoc Minh-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{We have } \forall\, m,n,p,u,v,w>0, (n+p)u+(p+m)v+(m+n)w\ge 2\sqrt{(mn+np+pm)(uv+vw+wu)}\to (1) \text{ and}, \sum_{cyc}\sqrt[3]{2\left(r_a^2+r_b^2\right)^2.r_c^2}=$$

$$(r_b^2+r_c^2).\sqrt[3]{\frac{2r_a^2}{r_b^2+r_c^2}}+(r_c^2+r_a^2).\sqrt[3]{\frac{2r_b^2}{r_c^2+r_a^2}}+(r_a^2+r_b^2).\sqrt[3]{\frac{2r_c^2}{r_a^2+r_b^2}}$$

$$=(n+p)u+(p+m)v+(m+n)w$$

$$\left(m=r_a^2, n=r_b^2, p=r_c^2 \text{ and } u=\sqrt[3]{\frac{2r_a^2}{r_b^2+r_c^2}}, v=\sqrt[3]{\frac{2r_b^2}{r_c^2+r_a^2}}, w=\sqrt[3]{\frac{2r_c^2}{r_a^2+r_b^2}}\right)$$

$$\overset{\text{via } (1)}{\ge} 2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\sum_{cyc}\left(\sqrt[3]{\frac{2r_a^2}{r_b^2+r_c^2}}.\sqrt[3]{\frac{2r_b^2}{r_c^2+r_a^2}}\right)}$$

$$=2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\sum_{cyc}\left(\sqrt[3]{\frac{2r_a^2}{r_c^2+r_a^2}.\frac{2r_b^2}{r_b^2+r_c^2}.1}\right)}\overset{\text{GM-HM}}{\ge}2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\sum_{cyc}\frac{3}{\frac{r_c^2+r_a^2}{2r_a^2}+\frac{r_b^2+r_c^2}{2r_b^2}+1}}$$

$$=2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\sum_{cyc}\frac{6r_a^2r_b^2}{r_b^2(r_c^2+r_a^2)+r_a^2(r_b^2+r_c^2)+2r_a^2r_b^2}}$$

$$=2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\sum_{cyc}\frac{6r_a^2r_b^2}{4r_a^2r_b^2+r_b^2r_c^2+r_c^2r_a^2}}\overset{\text{Bergstrom}}{\ge}2\sqrt{\sum_{cyc}r_a^2r_b^2}.\sqrt{\frac{6\left(\sum_{cyc}r_ar_b\right)^2}{6\sum_{cyc}r_a^2r_b^2}}$$

$$=2s^2\Rightarrow 3\sum_{cyc}\sqrt[3]{2\left(r_a^2+r_b^2\right)^2.r_c^2}\ge 6s^2=4s^2-16Rr+5r^2+2s^2+16Rr-5r^2$$

$$\overset{\substack{\text{Chu-Yang}\\ \text{and}\\ \text{Dang Ngoc Minh}}}{\ge}\left(\sum_{cyc}m_a\right)^2+\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}w_a\right)$$

(**Reference : Inequality in Triangle by Dang Ngoc Minh – 120; published at www.ssmrmh.ro**)

$$= \left(\sum_{\text{cyc}} \mathbf{m}_a\right)\left(\sum_{\text{cyc}} (\mathbf{m}_a + \mathbf{w}_a)\right)$$

$$\therefore 3\sum_{\text{cyc}} \sqrt[3]{2\left(\mathbf{r}_a^2 + \mathbf{r}_b^2\right)^2 . \mathbf{r}_c^2} \geq \left(\sum_{\text{cyc}} \mathbf{m}_a\right)\left(\sum_{\text{cyc}} (\mathbf{m}_a + \mathbf{w}_a)\right) \forall \Delta\, \mathbf{ABC},$$

$'' ='' $ **iff** $\Delta$ **ABC is equilateral (QED)**