ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC $a \neq b \neq c \neq a$,

O – circumcenter, I – incenter, H – orthocenter, R', s', r' are

the circumradius, semiperimeter, inradius of ΔOIH. Prove that

$$s' < \sqrt{2}R' + \left(\sqrt{2} - 1\right)r'$$

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We will first prove the lemma that $\widehat{OIH} > \frac{3\pi}{4}$.

$$\widehat{OIH} > \frac{3\pi}{4} \Leftrightarrow \cos \widehat{OIH} < -\frac{\sqrt{2}}{2} \Leftrightarrow \sqrt{2}.IO.IH < OH^2 - IO^2 - IH^2$$

$$\Leftrightarrow \sqrt{2R(R-2r)(4R^2+4Rr+3r^2-s^2)}$$

$$< \left(9R^2+8Rr+2r^2-2s^2\right) - R(R-2r) - \left(4R^2+4Rr+3r^2-s^2\right) = 4R^2+6Rr-r^2-s^2$$

Since $4R^2+6Rr-r^2-s^2 = \left(4R^2+4Rr+3r^2-s^2\right) + 2r(R-2r) > 0$, then after squaring we

get the equivalent inequality

$$2R(R-2r)\left(4R^2+4Rr+3r^2-s^2\right) < \left[\left(4R^2+4Rr+3r^2-s^2\right) + 2r(R-2r)\right]^2$$

$$\Leftrightarrow 0 < \left(4R^2+4Rr+3r^2-s^2\right)^2 + 2(R-2r)^2\left[s^2-(2R+r)^2\right],$$

which is true since

$s > 2R + r$ (Ciamberlini's inequality). So the proof of the lemma is complete.

Now, we have in any ΔABC, $s = a + (s-a) = 2R\sin A + r\cot\frac{A}{2}$, then, in ΔOIH, we have

$$s' = 2R'\sin\widehat{OIH} + r'\cot\frac{\widehat{OIH}}{2},$$

and since $x \to \sin x$ and $x \to \cot\frac{x}{2}$ are strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$, then

$$s' = 2R'\sin\widehat{OIH} + r'\cot\frac{\widehat{OIH}}{2} < 2R'\sin\frac{3\pi}{4} + r'\cot\frac{3\pi}{8} = \sqrt{2}R' + \left(\sqrt{2}-1\right)r'$$