ROMANIAN MATHEMATICAL MAGAZINE

**In $\Delta ABC$ can degenerate into a straight line, $k=\frac{43+4\sqrt{2}-\sqrt{344\sqrt{2}-235}}{46}$,**

**$m=\frac{1+32\sqrt{2}+\sqrt{64\sqrt{2}-67}}{46}$, the following relationship holds :**

$$m\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3(1-k)+k\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)$$

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Let $a:=y+z, b:=z+x, c:=x+y,\ \ x,y,z\geq 0$. WLOG, we assume that

$$z=min\{x,y,z\}, \text{then}$$

$\exists u,v\geq 0$ such that $x=u+z, y=v+z$, so $a=v+2z, b=u+2z, c=u+v+2z$.

$$m\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\Leftrightarrow m\left(a^2b+b^2c+c^2a\right)\geq ab^2+bc^2+ca^2$$

$$\Leftrightarrow m\left[(v+2z)^2(u+2z)+(u+2z)^2(u+v+2z)+(u+v+2z)^2(v+2z)\right]$$
$$\geq (v+2z)(u+2z)^2+(u+2z)(u+v+2z)^2+(u+v+2z)(v+2z)^2$$

$$\Leftrightarrow m\left[u^3+2u^2v+3uv^2+v^3+8(u+v)^2z+24(u+v)z^2+24z^3\right]$$
$$\geq u^3+3u^2v+2uv^2+v^3+8(u+v)^2z+24(u+v)z^2+24z^3,$$

Since $m>1$, so it suffices to prove that

$$m\left(u^3+2u^2v+3uv^2+v^3\right)\geq u^3+3u^2v+2uv^2+v^3,$$

This inequality is true for $v=0$.

Assume now that $v>0$. The last inequality is equivalent to

$$m\geq\frac{t^3+3t^2+2t+1}{t^3+2t^2+3t+1}=f(t), \text{where } t=\frac{u}{v}\geq 0. \qquad (1)$$

We have

$$f'(t)=\frac{-t^4+2t^3+5t^2+2t-1}{(t^3+2t^2+3t+1)^2}=\frac{-\left[t^2+\left(2\sqrt{2}-1\right)t+1\right]\left[t^2-\left(2\sqrt{2}+1\right)t+1\right]}{(t^3+2t^2+3t+1)^2}$$

$$=\frac{-\left[t^2+\left(2\sqrt{2}-1\right)t+1\right]\left(t-\frac{1+2\sqrt{2}-\sqrt{5+4\sqrt{2}}}{2}\right)\left(t-\frac{1+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{2}\right)}{(t^3+2t^2+3t+1)^2},$$

$$\Rightarrow \max_{t\in[0,\infty)} f(t) = \max\left\{f(0), f\left(\frac{1+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{2}\right)\right\} = \max\left\{1, \frac{1+32\sqrt{2}+\sqrt{64\sqrt{2}-67}}{46}\right\}$$
$$= m.$$

**which completes the proof of $(1)$.**

**Now, we have**

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3(1-k)+k\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right) \Leftrightarrow ab^2+bc^2+ca^2$$
$$\ge 3(1-k)abc + k\left(a^2b+b^2c+c^2a\right)$$

$$\Leftrightarrow (v+2z)(u+2z)^2+(u+2z)(u+v+2z)^2+(u+v+2z)(v+2z)^2$$
$$\ge 3(1-k)(v+2z)(u+2z)(u+v+2z)$$
$$+k\left[(v+2z)^2(u+2z)+(u+2z)^2(u+v+2z)+(u+v+2z)^2(v+2z)\right]$$

$$\Leftrightarrow u^3+3u^2v+2uv^2+v^3+8(u+v)^2z+24(u+v)z^2+24z^3$$
$$\ge 3(1-k)\left[u^2v+uv^2+2\left(u^2+3uv+v^2\right)z+8(u+v)z^2+8z^3\right]$$
$$+k\left[u^3+2u^2v+3uv^2+v^3+8(u+v)^2z+24(u+v)z^2+24z^3\right]$$

$$\Leftrightarrow u^3-uv^2+v^3+2\left(u^2-uv+v^2\right)z \ge k\left[u^3-u^2v+v^3+2\left(u^2-uv+v^2\right)z\right],$$

**Since $k<1$, so it suffices to prove that**

$$u^3-uv^2+v^3 \ge k\left(u^3-u^2v+v^3\right),$$

**This inequality is true for $v=0$. Assume now that $v > 0$. The last inequality is equivalent to**

$$k \le \frac{t^3-t+1}{t^3-t^2+1} = g(t), \text{where } t=\frac{u}{v}\ge 0. \qquad (2)$$

**We have**

$$g'(t) = \frac{-t^4+2t^3-t^2+2t-1}{(t^3-t^2+1)^2} = \frac{-\left[t^2+\left(\sqrt{2}-1\right)t+1\right]\left[t^2-\left(\sqrt{2}+1\right)t+1\right]}{(t^3-t^2+1)^2}$$

$$= \frac{-\left[t^2+\left(\sqrt{2}-1\right)t+1\right]\left(t-\frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}\right)\left(t-\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}\right)}{(t^3-t^2+1)^2},$$

$$\Rightarrow \min_{t\in[0,\infty)} g(t) = \min\left\{g\left(\frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}\right), \lim_{t\to\infty} g(t)\right\}$$
$$= \min\left\{\frac{43+4\sqrt{2}-\sqrt{344\sqrt{2}-235}}{46}, 1\right\} = k.$$

**which completes the proof of $(2)$. Equality holds iff $\Delta ABC$ is equilateral.**