ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta ABC, x, y, z > 0, -1 \le k \le 1$, the following relationship holds :

$$a(ka+b+c)x + b(kb+c+a)y + c(kc+a+b)z \ge 4F\sqrt{\left(k^2 + \frac{2(k+1)R}{r}\right)(xy+yz+zx)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove the following lemma, that for any $x, y, z, u, v, w > 0$, we have

$$(v+w)x + (w+u)y + (u+v)z \ge 2\sqrt{(uv+vw+wu)(xy+yz+zx)}.$$

By CBS inequality, we have

$$(v+w)x + (w+u)y + (u+v)z = (u+v+w)(x+y+z) - (ux+vy+wz)$$

$$= \sqrt{[2(uv+vw+wu) + u^2+v^2+w^2][2(xy+yz+zx) + x^2+y^2+z^2]} - (ux+vy+wz)$$

$$\ge 2\sqrt{(uv+vw+wu)(xy+yz+zx)} + ux+vy+wz - (ux+vy+wz)$$

$$= 2\sqrt{(uv+vw+wu)(xy+yz+zx)},$$

which completes the proof of the lemma. Equality holds iff $\frac{u}{x} = \frac{v}{y} = \frac{w}{z}$.

Now, for $u = \frac{k(b^2+c^2-a^2)+2bc}{2} = (1+k)s(s-a) + (1-k)(s-b)(s-c) > 0,$

$$v = \frac{k(c^2+a^2-b^2)+2ca}{2} > 0, w = \frac{k(a^2+b^2-c^2)+2ab}{2} > 0, \text{ we have}$$

$$a(ka+b+c)x + b(kb+c+a)y + c(kc+a+b)z = (v+w)x + (w+u)y + (u+v)z \ge$$

$$\ge 2\sqrt{(uv+vw+wu)(xy+yz+zx)},$$

with

$$4(uv+vw+wu) = \sum_{cyc}[k(c^2+a^2-b^2)+2ca][k(a^2+b^2-c^2)+2ab]$$

$$= k^2\sum_{cyc}(c^2+a^2-b^2)(a^2+b^2-c^2) + 2k\sum_{cyc} bc[(a^2+b^2-c^2)+(c^2+a^2-b^2)]$$

$$+ 4\sum_{cyc} a^2bc$$

$$= k^2.16F^2 + 2k.16s^2Rr + 4.8s^2Rr = 16F^2\left(k^2 + \frac{2(k+1)R}{r}\right),$$

So the proof is complete. Equality holds iff ΔABC is equilateral.