ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$n_b n_c \le (s-a)^2 + a^2\sqrt{1-\sin B\sin C}$$

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We have

$$n_a{}^2 = s\left(s-a+\frac{(b-c)^2}{a}\right) = s\left(s-\frac{4(s-b)(s-c)}{a}\right) = s^2 - \frac{4[a+(s-a)](s-b)(s-c)}{a}$$

$$= s^2 - 4(s-b)(s-c) - \frac{4sr^2}{a}.$$

Then

$$2n_b n_c \le n_b{}^2 + n_c{}^2 = \left(s^2 - 4(s-c)(s-a) - \frac{4sr^2}{b}\right) + \left(s^2 - 4(s-a)(s-b) - \frac{4sr^2}{c}\right)$$

$$= 2s^2 - 4a(s-a) - \frac{4sr^2(b+c)}{bc} = 2(s-a)^2 + 2a^2 - \frac{a^2bc(b+c)}{4sR^2},$$

$$\Rightarrow \frac{n_b n_c - (s-a)^2}{a^2} \le 1 - \frac{bc(b+c)}{8sR^2} = 1 - \frac{(b+c)\sin B\sin C}{2s} \overset{?}{\le} \sqrt{1-\sin B\sin C}$$

$$\Leftrightarrow \frac{(b+c)\sin B\sin C}{2s} \ge 1 - \sqrt{1-\sin B\sin C} = \frac{\sin B\sin C}{1+\sqrt{1-\sin B\sin C}}$$

$$\Leftrightarrow \sqrt{1-\sin B\sin C} \ge \frac{2s}{b+c} - 1 = \frac{a}{b+c}.$$

which is true because

$$\sqrt{1-\sin B\sin C} = \sqrt{1-\frac{h_a{}^2}{bc}} \ge \sqrt{1-\frac{w_a{}^2}{bc}} = \sqrt{1-\frac{4s(s-a)}{(b+c)^2}} = \frac{a}{b+c}.$$

The proof is complete. Equality holds iff $b=c$.