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In any ΔABC, the following relationship holds :

$$\frac{1}{\left(\frac{R}{a}\right)^{2R+r}+\left(\frac{R}{a}\right)^{2r+R}+1}+\frac{1}{\left(\frac{s}{b}\right)^{2R+r}+\left(\frac{s}{b}\right)^{2r+R}+1}+\frac{1}{\left(\frac{4r}{c}\right)^{2R+r}+\left(\frac{4r}{c}\right)^{2r+R}+1}\geq 1$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $u:=\frac{R}{a}, v:=\frac{s}{b}, w:=\frac{4r}{c}$. We have $uvw=1$.

We have $u^{2R+r}+u^{2r+R}=u^{2(R+r)}+u^{R+r}-u^{R+r}\left(u^R-1\right)(u^r-1)\leq u^{2(R+r)}+u^{R+r}$,

because u^R-1, u^r-1 have the same sign.

And since $u^{R+r}.v^{R+r}.w^{R+r}=1$, then $\exists x,y,z>0$ such that

$$u^{R+r}=\frac{yz}{x^2}, v^{R+r}=\frac{zx}{y^2}, w^{R+r}=\frac{xy}{z^2}.$$

Therefore

$$\sum_{cyc}\frac{1}{\left(\frac{R}{a}\right)^{2R+r}+\left(\frac{R}{a}\right)^{2r+R}+1}=\sum_{cyc}\frac{1}{u^{2R+r}+u^{2r+R}+1}\geq\sum_{cyc}\frac{1}{u^{2(R+r)}+u^{R+r}+1}=$$

$$=\sum_{cyc}\frac{x^4}{(yz)^2+x^2yz+x^4}\overset{AM-GM}{\geq}\sum_{cyc}\frac{x^4}{(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4}$$

$$\overset{CBS}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}\left[(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4\right]}=1.$$