ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{2a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\geq\frac{2a^2b}{c}+\frac{b^2c}{a}+\frac{a^2c}{b}+b(c-b)$$

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$$a^4b+b^4c+c^4a\overset{?}{\geq}a^3b^2+b^3c^2+c^3a^2$$

$$\Leftrightarrow(z+x)(y+z)^4+(x+y)(z+x)^4+(y+z)(x+y)^4\overset{?}{\geq}$$
$$(z+x)^2(y+z)^3+(x+y)^2(z+x)^3+(y+z)^2(x+y)^3$$

$$(x=s-a,y=s-b,z=s-c)\Leftrightarrow\sum_{cyc}xy^4+\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3\overset{?}{\geq}3xyz\sum_{cyc}xy$$

$$\to\text{true}\because\sum_{cyc}xy^4+\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3\overset{\text{AM-GM}}{\geq}3\sum_{cyc}x^2y^3=3xyz\sum_{cyc}\frac{xy^2}{z}$$

$$=3xyz\sum_{cyc}\frac{x^2y^2}{zx}\overset{\text{Bergstrom}}{\geq}3xyz.\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}xy}=3xyz\sum_{cyc}xy$$

$$\therefore a^4b+b^4c+c^4a\geq a^3b^2+b^3c^2+c^3a^2\Rightarrow\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\geq\frac{a^2b}{c}+\frac{b^2c}{a}+\frac{c^2a}{b}$$

$$\text{and so, it remains to prove}:\frac{a^3}{c}\overset{?}{\geq}\frac{a^2b}{c}-\frac{c^2a}{b}+\frac{a^2c}{b}+b(c-b)$$

$$\Leftrightarrow a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\overset{?}{\geq}0$$

$$\Leftrightarrow(y-x)(z+x)(y+z)^2+(z-y)(x+y)(z+x)^2+(x-z)(y+z)(x+y)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\sum_{cyc}xy^3\overset{?}{\geq}xyz\sum_{cyc}x\to\text{true}\because\sum_{cyc}xy^3=xyz\sum_{cyc}\frac{y^2}{z}\overset{\text{Bergstrom}}{\geq}xyz.\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x}$$

$$=xyz\sum_{cyc}x\therefore\frac{2a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\geq\frac{2a^2b}{c}+\frac{b^2c}{a}+\frac{a^2c}{b}+b(c-b)\ \forall\ ABC,$$

$$''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral}\ (QED)$$