ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with M being an arbitrary point in the plane of the triangle, the following relationship holds :

$$\mathrm{MA} + \mathrm{MB} + \mathrm{MC} \geq 2\sqrt{(4\mathrm{R} + \mathrm{r})\mathrm{r}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

For simplicity, we shall denote MA by R_1, MB by R_2, MC by R_3 and firstly, we have that : if $F(a, b, c, R_1, R_2, R_3) \geq 0$, then via inversion transformation, $F(aR_1, bR_2, cR_3, R_2R_3, R_3R_1, R_1R_2) \geq 0 \to (1)$ and when M is situated on one of the vertices of Δ ABC (WLOG on the vertex A, say), then : $(\mathrm{MA} + \mathrm{MB} + \mathrm{MC})^2 =$

$$(R_2 + R_3)^2 = (b + c)^2 \ (\because R_2 = c, R_3 = b) \overset{?}{\geq} 4r(4R + r) = 2\sum_{cyc} ab - \sum_{cyc} a^2$$

$$\Leftrightarrow a^2 + 2(b^2 + c^2) - 2a(b + c) \overset{?}{\geq} 0 \Leftrightarrow a^2 + (b + c)^2 - 2a(b + c) + (b - c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (b + c - a)^2 + (b - c)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \mathrm{MA} + \mathrm{MB} + \mathrm{MC} > 2\sqrt{(4R + r)r}$$

and we now focus on the case when M is not situated on any of the vertices of Δ ABC and now, via Klamkin's Polar Moment of Inertia, $\left(xR_1^2 + yR_2^2 + zR_3^2\right)\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) \geq$

$$\frac{a^2}{x} + \frac{y^2}{b} + \frac{z^2}{c} \overset{\text{Bergstrom}}{\geq} \frac{(a + b + c)^2}{x + y + z} \overset{\text{via } (1)}{\Rightarrow}$$

$$\left(xR_2^2R_3^2 + yR_3^2R_1^2 + zR_1^2R_2^2\right)\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) \geq \frac{(aR_1 + bR_2 + cR_3)^2}{x + y + z} \Rightarrow$$

$$\frac{R_2^2R_3^2}{yz} + \frac{R_3^2R_1^2}{zx} + \frac{R_1^2R_2^2}{xy} \geq \left(\frac{aR_1 + bR_2 + cR_3}{x + y + z}\right)^2 \Rightarrow \frac{R_2^2R_3^2}{yR_2^2 . zR_3^2} + \frac{R_3^2R_1^2}{zR_3^2 . xR_1^2} + \frac{R_1^2R_2^2}{xR_1^2 . yR_2^2} \geq$$

$$\left(\frac{aR_1 + bR_2 + cR_3}{xR_1^2 + yR_2^2 + zR_3^2}\right)^2 \ \left(\text{via } x \equiv xR_1^2, y \equiv yR_2^2, z \equiv zR_3^2\right)$$

$$\Rightarrow \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \geq \left(\frac{aR_1 + bR_2 + cR_3}{xR_1^2 + yR_2^2 + zR_3^2}\right)^2 \Rightarrow yz + zx + xy \geq \left(\frac{aR_1 + bR_2 + cR_3}{\frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z}}\right)^2$$

$$\left(\text{via } x \equiv \frac{1}{x}, y \equiv \frac{1}{y}, z \equiv \frac{1}{z}\right) \Rightarrow \frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{aR_1 + bR_2 + cR_3}{\sqrt{xy + yz + zx}}$$

$$\Rightarrow R_1 + R_2 + R_3 \geq \frac{aR_1 + bR_2 + cR_3}{\sqrt{R_1R_2 + R_2R_3 + R_3R_1}} \ (\text{via } x \equiv R_1, y \equiv R_2, z \equiv R_3)$$

$$\Rightarrow (R_1 + R_2 + R_3).\sqrt{R_1R_2 + R_2R_3 + R_3R_1} \geq aR_1 + bR_2 + cR_3$$

$$\overset{\text{Oppenheim}}{\geq} \sqrt{2\sum_{cyc} ab - \sum_{cyc} a^2} . \sqrt{R_1R_2 + R_2R_3 + R_3R_1} \ (\because \sqrt{a}, \sqrt{b}, \sqrt{c} \text{ form sides of a triangle})$$

$$\Rightarrow \mathbf{R_1 + R_2 + R_3 \geq \sqrt{2\sum_{cyc} ab - \sum_{cyc} a^2} = \sqrt{4r(4R + r)}}$$

$$\therefore \mathbf{MA + MB + MC \geq 2\sqrt{(4R + r)r}}$$

∀ ABC with M being *an arbitrary* point in the p*l*ane of the tri*angle* (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$\sqrt{a}, \sqrt{b}, \sqrt{c}$ can be the sides of a triangle Δ' with area F' such that

$$4F' = \sqrt{2(ab + bc + ca) - (a^2 + b^2 + c^2)} = 2\sqrt{(4R + r)r}.$$

Using Oppenheim's inequality in triangle Δ', we have

$$MA + MB + MC = \frac{MA}{a}.\sqrt{a}^2 + \frac{MB}{b}.\sqrt{b}^2 + \frac{MC}{c}.\sqrt{c}^2 \geq$$

$$\geq 4F'\sqrt{\frac{MA.MB}{ab} + \frac{MB.MC}{bc} + \frac{MC.MA}{ca}} \overset{Hayashi}{\geq} 4F' =$$

$$= 2\sqrt{(4R + r)r}.$$