ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with M being an arbitrary point in the plane of the triangle, the following relationship holds :

$$(MA+MB+MC)\sqrt{MA\cdot MB+MB\cdot MC+MC\cdot MA}\geq 4F$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

For simplicity, we shall denote MA by R_1, MB by R_2, MC by R_3 and firstly, we have that : if $F(a,b,c,R_1,R_2,R_3)\geq 0$, then via inversion transformation, $F(aR_1,bR_2,cR_3,R_2R_3,R_3R_1,R_1R_2)\geq 0 \to ①$ and when M is situated on one of the vertices of Δ ABC (WLOG on the vertex A, say), then :

$$(MA+MB+MC).\sqrt{MA.MB+MB.MC+MC.MA}=(R_2+R_3).\sqrt{R_2R_3}\overset{AM-GM}{\geq}$$

$$2R_2R_3=2bc\ (\because R_2=c,R_3=b)\overset{?}{\geq}4F=\frac{4abc}{4R}\Leftrightarrow 2R\overset{?}{\geq}2R.\sin A\Leftrightarrow \sin A\overset{?}{\leq}1$$

$\to$ true $\therefore (MA+MB+MC).\sqrt{MA.MB+MB.MC+MC.MA}\geq 4F$ and we now focus on the case when M is not situated on any of the vertices of Δ ABC and we have, via Klamkin's Polar Moment of Inertia, $\left(xR_1^2+yR_2^2+zR_3^2\right)\left(\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}\right)\geq$

$$\frac{a^2}{x}+\frac{y^2}{b}+\frac{z^2}{c}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{x+y+z}\overset{via\ ①}{\Rightarrow}$$

$$\left(xR_2^2R_3^2+yR_3^2R_1^2+zR_1^2R_2^2\right)\left(\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}\right)\geq\frac{(aR_1+bR_2+cR_3)^2}{x+y+z}$$

$$\Rightarrow\frac{R_2^2R_3^2}{yz}+\frac{R_3^2R_1^2}{zx}+\frac{R_1^2R_2^2}{xy}\geq\left(\frac{aR_1+bR_2+cR_3}{x+y+z}\right)^2\Rightarrow\frac{R_2^2R_3^2}{yR_2^2.zR_3^2}+\frac{R_3^2R_1^2}{zR_3^2.xR_1^2}+\frac{R_1^2R_2^2}{xR_1^2.yR_2^2}$$

$$\geq\left(\frac{aR_1+bR_2+cR_3}{xR_1^2+yR_2^2+zR_3^2}\right)^2\ (\text{via } x\equiv xR_1^2,y\equiv yR_2^2,z\equiv zR_3^2)$$

$$\Rightarrow\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}\geq\left(\frac{aR_1+bR_2+cR_3}{xR_1^2+yR_2^2+zR_3^2}\right)^2\Rightarrow yz+zx+xy\geq\left(\frac{aR_1+bR_2+cR_3}{\frac{R_1^2}{x}+\frac{R_2^2}{y}+\frac{R_3^2}{z}}\right)^2$$

$$\left(\text{via } x\equiv\frac{1}{x},y\equiv\frac{1}{y},z\equiv\frac{1}{z}\right)\Rightarrow\frac{R_1^2}{x}+\frac{R_2^2}{y}+\frac{R_3^2}{z}\geq\frac{aR_1+bR_2+cR_3}{\sqrt{xy+yz+zx}}$$

$$\Rightarrow R_1+R_2+R_3\geq\frac{aR_1+bR_2+cR_3}{\sqrt{R_1R_2+R_2R_3+R_3R_1}}\ (\text{via } x\equiv R_1,y\equiv R_2,z\equiv R_3)$$

$$\Rightarrow (MA+MB+MC).\sqrt{MA.MB+MB.MC+MC.MA}\geq aMA+bMB+cMC$$

$$\overset{well-known}{\geq}4F\ \therefore\ \text{combining both cases,}$$

$$(MA+MB+MC).\sqrt{MA.MB+MB.MC+MC.MA}\geq 4F\ \forall\ ABC$$

with M being an arbitrary point in the plane of the triangle (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

From the polar moment of inertia inequality (Klamkin's inequality), we have,

for any $x, y, z \in \mathbb{R}$,

$$(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2.$$

For $x = MB.MC, y = MC.MA, z = MA.MB$, we get

$$(MA.MB+MB.MC+MC.MA)(MA+MB+MC) \geq MA.a^2+MB.b^2+MC.c^2.$$

Also by Oppenheim's inequality, we have

$$MA.a^2+MB.b^2+MC.c^2 \geq 4F.\sqrt{MA.MB+MB.MC+MC.MA}.$$

Using these two inequalities, we get

$$(MA+MB+MC)\sqrt{MA.MB+MB.MC+MC.MA} \geq 4F,$$

as desired.