ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $x = \sum_{cyc} r_a, y = \sum_{cyc} n_a, z = \sum_{cyc} h_a$, the following relationship holds :

$$\frac{1}{\left(\frac{x}{a}\right)^2\left(\frac{y}{b}+\frac{z}{c}\right)+\frac{81\sqrt{3}}{8}}+\frac{1}{\left(\frac{y}{b}\right)^2\left(\frac{z}{c}+\frac{x}{a}\right)+\frac{81\sqrt{3}}{8}}+\frac{1}{\left(\frac{z}{c}\right)^2\left(\frac{x}{a}+\frac{y}{b}\right)+\frac{81\sqrt{3}}{8}} \le \frac{8}{81\sqrt{3}}$$

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$$\text{We shall first prove that : } \frac{1}{abc}\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} n_a\right)\left(\sum_{cyc} h_a\right) \overset{?}{\ge} \frac{81\sqrt{3}}{8}$$

$$\Leftrightarrow \frac{(4R+r)^2}{16R^2r^2s^2}\cdot\left(\sum_{cyc} n_a\right)^2 \cdot \frac{(s^2+4Rr+r^2)^2}{4R^2} \overset{?}{\ge} \frac{19683}{64} \;\&$$

$$\because \left(\sum_{cyc} n_a\right)^2 \overset{\text{Ben Ajiba}}{\ge} 9s^2-80Rr-2r^2$$

$$\left(\begin{gathered}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 74;\\ \text{published at www.ssmrmh.ro}\end{gathered}\right)$$

$$\therefore \text{ it suffices to prove :}$$

$$(4R+r)^2(9s^2-80Rr-2r^2)(s^2+4Rr+r^2)^2 \overset{?}{\ge} 19683R^4r^2s^2$$

$$\Leftrightarrow (144R^2+72Rr+9r^2)s^6 - r(128R^3-192R^2r-120Rr^2-16r^3)s^4 -$$
$$r^2(27619R^4+5632R^3r+1248R^2r^2+64Rr^3-5r^4)s^2 -$$
$$r^3(20480R^5+20992R^4r+8192R^3r^2+1472R^2r^3+112Rr^4+2r^5) \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

$$\text{and } \because P = (144R^2+72Rr+9r^2)(s^2-16Rr+5r^2)^3 +$$
$$r(6784R^3+1488R^2r-528Rr^2-119r^3)(s^2-16Rr+5r^2)^2 +$$
$$r^2(78877R^4-12032R^3r-1617R^2r^2+328Rr^3+520r^4)(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow 94672t^5-161425t^4-70016t^3+27280t^2+2528t-752 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(94672t^4+27919t^3-14178t^2-1076t+376) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{x}{a}.\frac{y}{b}.\frac{z}{c} \geq \frac{81\sqrt{3}}{8} \Rightarrow XYZ \geq \frac{1}{p}\left(X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c}, P = \frac{8}{81\sqrt{3}}\right)$$

$$\Rightarrow XYZp \geq 1 \rightarrow (1) \;\&\; \frac{1}{\left(\frac{x}{a}\right)^2\left(\frac{y}{b}+\frac{z}{c}\right)+\frac{81\sqrt{3}}{8}} + \frac{1}{\left(\frac{y}{b}\right)^2\left(\frac{z}{c}+\frac{x}{a}\right)+\frac{81\sqrt{3}}{8}} + \frac{1}{\left(\frac{z}{c}\right)^2\left(\frac{x}{a}+\frac{y}{b}\right)+\frac{81\sqrt{3}}{8}}$$

$$\leq \frac{8}{81\sqrt{3}} \Leftrightarrow \frac{1}{pX^2(Y+Z)+1} + \frac{1}{pY^2(Z+X)+1} + \frac{1}{pZ^2(X+Y)+1} \leq 1$$

$$\overset{\text{clearing denominators and simplifying}}{\Leftrightarrow} 2(p^3X^3Y^3Z^3 - 1)$$

$$(pX^2Y + pX^2Z + pXY^2 + pXZ^2 + pY^2Z + pYZ^2)(p^2X^2Y^2Z^2 - 1) \geq 0 \rightarrow \text{true}$$

$$\because XYZp \overset{\text{via } (1)}{\geq} 1 \therefore \frac{1}{\left(\frac{x}{a}\right)^2\left(\frac{y}{b}+\frac{z}{c}\right)+\frac{81\sqrt{3}}{8}} + \frac{1}{\left(\frac{y}{b}\right)^2\left(\frac{z}{c}+\frac{x}{a}\right)+\frac{81\sqrt{3}}{8}} + \frac{1}{\left(\frac{z}{c}\right)^2\left(\frac{x}{a}+\frac{y}{b}\right)+\frac{81\sqrt{3}}{8}}$$

$$\leq \frac{8}{81\sqrt{3}} \;\forall\, ABC \text{ with } x = \sum_{cyc} r_a, y = \sum_{cyc} n_a, z = \sum_{cyc} h_a,$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$