ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC, the $following$ $relationship$ $holds$:

$$\frac{g_a^{g_a}}{g_a^2+3}+\frac{g_b^{g_b}}{g_b^2+3}+\frac{g_c^{g_c}}{g_c^2+3}\ge\frac{9}{16}\left(1+\frac{r_ar_br_c}{n_a+n_b+n_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove that}: x^x\ge\frac{(x^2+3)^2}{16}\ \forall\, x>0\to(1)\text{ and }(1)\Leftrightarrow$$

$$x.\ln x\ge 2\ln\frac{x^2+3}{4}\to(2)\ \&\text{ we shall now prove}:\ln x\ge 1-\frac{1}{x}\ \forall\, x>0\text{ and}$$

$$\text{if }F(x)=\ln x-1+\frac{1}{x},\text{ then}: F'(x)=\frac{x-1}{x^2}\le 0\ \forall\, x\in(0,1]\Rightarrow F(x)\text{ is}\downarrow\text{on }(0,1]$$

$$\Rightarrow F(x)\ge F(1)=0\Rightarrow\ln x\ge 1-\frac{1}{x}\ \forall\, x\in(0,1]\text{ and }\forall\, x\ge 1, F'(x)\ge 0$$

$$\Rightarrow F(x)\text{ is}\uparrow\forall\, x\ge 1\Rightarrow F(x)\ge F(1)=0\Rightarrow\ln x\ge 1-\frac{1}{x}\ \forall\, x\ge 1\text{ and so,}$$

$$\ln x\ge 1-\frac{1}{x}\ \forall\, x>0\to(\bullet)$$

$$\text{Let }f(x)=x.\ln x-2\ln\frac{x^2+3}{4}\ \forall\, x>0\text{ and then}: f'(x)=\ln x-\frac{4x}{x^2+3}+1$$

$$\boxed{\text{Case 1}}\ 0<x\le 1\text{ and then}:\ln x-\frac{4x}{x^2+3}+1\le x-1-\frac{4x}{x^2+3}+1$$

$$=\frac{x(x^2-1)}{x^2+3}\le 0\Rightarrow f'(x)\le 0\Rightarrow f(x)\text{ is}\downarrow\text{on }(0,1]\Rightarrow f(x)\ge f(1)=0$$

$$\therefore x.\ln x\ge 2\ln\frac{x^2+3}{4}\ \forall\, x\in(0,1]$$

$$\boxed{\text{Case 2}}\ x\ge 1\text{ and then}:\ln x-\frac{4x}{x^2+3}+1\ge\frac{x-1}{x}-\frac{4x}{x^2+3}+1$$

$$=\frac{(x-1)(2x^2-3x+3)}{x(x^2+3)}=\frac{(x-1)(2(x-1)^2+x+1)}{x(x^2+3)}\ge 0\Rightarrow f'(x)\ge 0$$

$$\Rightarrow f(x)\text{ is}\uparrow\forall\, x\ge 1\Rightarrow f(x)\ge f(1)=0\ \therefore x.\ln x\ge 2\ln\frac{x^2+3}{4}\ \forall\, x\ge 1$$

$$\text{and so, }x.\ln x\ge 2\ln\frac{x^2+3}{4}\ \forall\, x>0\Rightarrow(2)\Rightarrow(1)\text{ is true}\overset{\text{via }(1)}{\Rightarrow}\frac{g_a^{g_a}}{g_a^2+3}+$$

$$\frac{g_b^{g_b}}{g_b^2+3}+\frac{g_c^{g_c}}{g_c^2+3}\geq\sum_{cyc}\frac{\frac{(g_a^2+3)^2}{16}}{g_a^2+3}=\frac{1}{16}\left(\sum_{cyc}g_a^2+9\right)\overset{?}{\geq}\frac{9}{16}\left(1+\frac{r_a r_b r_c}{n_a+n_b+n_c}\right)$$

$$\Leftrightarrow\sum_{cyc}g_a^2\overset{?}{\geq}\frac{9r_a r_b r_c}{n_a+n_b+n_c}\Leftrightarrow\left(\sum_{cyc}g_a^2\right)^2\left(\sum_{cyc}n_a\right)^2\underset{(*)}{\overset{?}{\geq}}81r^2s^4$$

$$\text{Now,}\left(\sum_{cyc}g_a^2\right)^2\left(\sum_{cyc}n_a\right)^2\overset{\text{Bogdan Fustei}}{=}\left(\sum_{cyc}((s-a)^2+2r.h_a)\right)^2\left(\sum_{cyc}n_a\right)^2$$

$$=\left(\sum_{cyc}(s-a)^2+\frac{r(s^2+4Rr+r^2)}{R}\right)^2\left(\sum_{cyc}n_a\right)^2$$

$$=\frac{\left(R(s^2-8Rr-2r^2)+r(s^2+4Rr+r^2)\right)^2}{R^2}.\left(\sum_{cyc}n_a\right)^2\overset{\text{Ben Ajiba}}{\geq}$$

$$\frac{\left(R(s^2-8Rr-2r^2)+r(s^2+4Rr+r^2)\right)^2(9s^2-80Rr-2r^2)}{R^2}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba}-74;\\ \text{published at www.ssmrmh.ro}\end{array}\right)\overset{?}{\geq}81r^2s^4$$

$$\Leftrightarrow 9(R+r)^2s^6-r(224R^3+351R^2r+30Rr^2-16r^3)s^4+$$

$$r^2(1856R^4+704R^3r-564R^2r^2-136Rr^3+5r^4)-$$

$$r^3(5120R^5-2432R^4r-1024R^3r^2+296R^2r^3+88Rr^4+2r^5)\boxed{\underset{(*)}{\overset{?}{\geq}}}0\ \text{and}\because P=$$

$$9(R+r)^2(s^2-16Rr+5r^2)^3+$$

$$r(208R^3+378R^2r+132Rr^2-119r^3)(s^2-16Rr+5r^2)^2+$$

$$r^2(1600R^4+1216R^3r+933R^2r^2-2294Rr^3+520r^4)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}P$$

$$\Leftrightarrow 512t^5-1304t^4+1238t^3-1702t^2+739t-94\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(512t^4-280t^3+678t^2-346t+47)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (*) \textbf{ is true} \therefore \frac{g_a^{g_a}}{g_a^2+3}+\frac{g_b^{g_b}}{g_b^2+3}+\frac{g_c^{g_c}}{g_c^2+3} \geq \frac{9}{16}\left(1+\frac{r_a r_b r_c}{n_a+n_b+n_c}\right)$$

$\forall$ **ABC, with equality iff Δ ABC is equilateral (QED)**