ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a^2\right) \le \left(\sum_{cyc} r_a\right)\left(\sum_{cyc} h_a^2\right) \le \left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a^2\right)$$

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$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a^2\right) \le \frac{s^2+4Rr+r^2}{2R}\cdot\frac{9s^2-16Rr+32r^2}{9}$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 112; published at www.ssmrmh.ro)

$$\overset{?}{\le} (4R+r)\cdot\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{4R^2}$$

$$\Leftrightarrow (18R+9r)s^4 - r(328R^2+82Rr-18r^2)s^2 +$$

$$r^2(704R^3+208R^2r+44Rr^2+9r^3) \boxed{\overset{?}{\underset{(*)}{\ge}}} 0 \ \& \because P = (18R+9r)(s^2-16Rr+5r^2)^2$$

$$+2r(124R^2+13Rr-36r^2)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P \Leftrightarrow 8t^3-5t^2-31t+18 \overset{?}{\ge} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(8t^2+11t-9) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a^2\right) \le \left(\sum_{cyc} r_a\right)\left(\sum_{cyc} h_a^2\right) \text{ and}$$

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc} h_a^2\right) \le \left(\sum_{cyc} h_a\right)\left(\sum_{cyc} r_a^2\right)$$

$$\Leftrightarrow (4R+r)\cdot\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{4R^2} \le \frac{s^2+4Rr+r^2}{2R}\cdot\left((4R+r)^2-2s^2\right)$$

$$\Leftrightarrow -(8R+r)s^4 + (32R^3+32R^2r-2Rr^2-2r^3)s^2$$

$$+r(128R^4+32R^3r-24R^2r^2-10Rr^3-r^4) \boxed{\overset{?}{\underset{(**)}{\ge}}} 0 \text{ and} \because Q =$$

$$-(8R+r)(s^2-4R^2-4Rr-3r^2) - r(4R^2+30Rr+5r^2)(s^2-4R^2-4Rr-3r^2)$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\ge} Q$$

$$\Leftrightarrow 14t^4-13t^3-22t^2-15t-2 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(14t^3+15t^2+8t+1) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true} \therefore \left(\sum_{\text{cyc}} r_a\right)\left(\sum_{\text{cyc}} h_a^2\right) \leq \left(\sum_{\text{cyc}} h_a\right)\left(\sum_{\text{cyc}} r_a^2\right)$$

$$\text{and so,}: \left(\sum_{\text{cyc}} h_a\right)\left(\sum_{\text{cyc}} w_a^2\right) \leq \left(\sum_{\text{cyc}} r_a\right)\left(\sum_{\text{cyc}} h_a^2\right) \leq \left(\sum_{\text{cyc}} h_a\right)\left(\sum_{\text{cyc}} r_a^2\right)$$

$\forall$ ABC, with equality iff Δ ABC is equilateral (QED)