ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$\sum_{cyc}\frac{1}{r_a^3+r_b^3+3s^2r}\ge\frac{9}{5s^2(2R-r)}$$

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For convenience, $s'\to$ semiperimeter, $R'\to$ circumradius & $r'\to$ inradius $\therefore$ we are to prove : $\sum_{cyc}\frac{1}{r_a^3+r_b^3+3s'^2r'}\ge\frac{9}{5s'^2(2R'-r')}$

We prove : $\sum_{cyc}\frac{1}{\alpha^3+3\alpha\beta\gamma+\beta^3}\ge\frac{18}{5\left(\sum_{cyc}\alpha^2\beta+\sum_{cyc}\alpha\beta^2\right)}\ \forall\ \alpha,\beta,\gamma>0;$

assigning $\beta+\gamma=x,\gamma+\alpha=y,\alpha+\beta=z\Rightarrow x+y-z=2\gamma>0,y+z-x=2\alpha>0$ and $z+x-y=2\beta>0\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius & inradius $=s,R,r$ (say); $\sum_{cyc}\alpha=s,\alpha\beta\gamma=r^2s,$

$\sum_{cyc}\alpha\beta=4Rr+r^2,\sum_{cyc}\alpha^3=s^3-12Rrs,\sum_{cyc}\alpha^3\beta^3=(4Rr+r^2)^3-12Rr^3s^2;$ with $m=s^3-12Rrs$, LHS $=\sum_{cyc}\frac{(\beta^3+3\alpha\beta\gamma+\gamma^3)(\gamma^3+3\alpha\beta\gamma+\alpha^3)}{(\alpha^3+3\alpha\beta\gamma+\beta^3)(\beta^3+33\alpha\beta\gamma+\gamma^3)(\gamma^3+3\alpha\beta\gamma+\alpha^3)}$

$$=\frac{\left(\sum_{cyc}\alpha^3\right)^2+\sum_{cyc}\alpha^3\beta^3+12\alpha\beta\gamma\sum_{cyc}\alpha^3+27\alpha^2\beta^2\gamma^2}{3\alpha\beta\gamma\left(\sum_{cyc}\alpha^3\right)^2+\left(\sum_{cyc}\alpha^3\right)\left(\sum_{cyc}\alpha^3\beta^3\right)+18\alpha^2\beta^2\gamma^2\sum_{cyc}\alpha^3+3\alpha\beta\gamma\sum_{cyc}\alpha^3\beta^3+26\alpha^3\beta^3\gamma^3}$$

$$\frac{m^2+(4Rr+r^2)^3-12Rr^3s^2+12r^2sm+27r^4s^2}{(3r^2sm+18r^4s^2)m+(m+3r^2s)((4Rr+r^2)^3-12Rr^3s^2)+26r^6s^3}\overset{?}{\ge}\frac{18/5}{\sum_{cyc}\alpha^2\beta+\sum_{cyc}\alpha\beta^2}$$

$=\frac{18}{5(s(4Rr+r^2)-3r^2s)}\Leftrightarrow(5R-16r)s^6-r(120R^2-498Rr+111r^2)s^4+r^2(432R^3-3948R^2r+1605Rr^2-189r^3)s^2+$

$r^3(3776R^4+1808R^3r-60R^2r^2-133Rr^3-16r^4)\boxed{\underset{(*)}{\overset{?}{\ge}}}0$ and $\because P=$

$(5R-10r)(s^2-16Rr+5r^2)^3-6rs^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+r(96R^2-177Rr+51r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen,Double Rouche}}{\ge}0\ \therefore$ to prove $(*)$,

it suffices to prove : LHS of $(*)\overset{?}{\ge}P\Leftrightarrow(48R^3-204R^2r-96Rr^2+57r^3)s^2$

$\underset{(**)}{\overset{?}{\ge}}r(320R^4-2320R^3r-564R^2r^2+173Rr^3+41r^4)\because r^3s^2\overset{\text{Gerretsen}}{\ge}16Rr^4-5r^5$

$\therefore$ LHS of $(**)\ge4(12R^3-51R^2r-24Rr^2+14r^3)+16Rr^4-5r^5\overset{?}{\ge}$ RHS of $(**)$

$\Leftrightarrow4(12R^3-51R^2r-24Rr^2+14r^3)s^2+3r^4(R-2r)$

$\overset{?}{\ge}4r(80R^4-580R^3r-141R^2r^2+40Rr^3+10r^4)$ and $\because 3r^4(R-2r)\overset{\text{Euler}}{\ge}0$

$$\therefore \text{ to prove } (**), \text{it suffices to prove : } (12R^3 - 51R^2r - 24Rr^2 + 14r^3)s^2$$

$$\boxed{\underset{(1)}{\overset{?}{\geq}}} r(80R^4 - 580R^3r - 141R^2r^2 + 40Rr^3 + 10r^4)$$

$\boxed{Case\ 1}$ $12R^3 - 51R^2r - 24Rr^2 + 14r^3 \geq 0$ *and then* : LHS of (1) $\overset{\text{Gerretsen}}{\geq}$

$$(12R^3 - 51R^2r - 24Rr^2 + 14r^3)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (1)$$

$$\Leftrightarrow 28t^4 - 74t^3 + 3t^2 + 76t - 20 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)\left((t-2)(28t^2 + 38t + 43) + 96\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (1) \text{ is true}$$

$\boxed{Case\ 2}$ $12R^3 - 51R^2r - 24Rr^2 + 14r^3 < 0$ *and then* : LHS of (1) $\overset{\text{Rouche}}{\geq}$

$$(12R^3 - 51R^2r - 24Rr^2 + 14r^3)\left(2R^2 + 10Rr - r^2 + 2(R-2r).\sqrt{R^2 - 2Rr}\right) \overset{?}{\geq}$$

$$\text{RHS of } (1) \Leftrightarrow 2(R-2r)(12R^4 - 7R^3r - 9R^2r^2 - 28Rr^3 + 6r^4) \overset{?}{\geq}$$

$$2(R-2r).\sqrt{R^2 - 2Rr}.\left(-(12R^3 - 51R^2r - 24Rr^2 + 14r^3)\right) \Leftrightarrow$$

$$(12R^4 - 7R^3r - 9R^2r^2 - 28Rr^3 + 6r^4)^2 \overset{?}{\geq} (R^2 - 2Rr)\begin{pmatrix} 12R^3 - 51R^2r - \\ 24Rr^2 + 14r^3 \end{pmatrix}^2$$

$$\Leftrightarrow 1344t^7 - 4640t^6 + 720t^5 + 7037t^4 - 612t^3 - 864t^2 + 56t + 36 \overset{?}{\geq} 0 \text{ and}$$

$$\because t^4 \overset{\text{Euler}}{\geq} 16 \therefore \text{ it suffices to prove :}$$

$$336t^7 - 1160t^6 + 180t^5 + 1759t^4 - 153t^3 - 216t^2 + 14t + 13 \underset{(2)}{\overset{?}{\geq}} 0 \text{ and } \because Q =$$

$$336(t-2)^7 + 3544(t-2)^6 + 14484(t-2)^5 + 28039(t-2)^4 \overset{\text{Euler}}{\geq} 0$$

$$\therefore \text{ in order to prove } (2), \text{it suffices to prove : LHS of } (2) \overset{?}{\geq} Q$$

$$\Leftrightarrow 23679t^3 - 139200t^2 + 268462t - 168931 \underset{(\bullet)}{\overset{?}{\geq}} 0 \Leftrightarrow$$

$$\frac{1}{125}\left((118395t - 175062)(5t - 11)^2 + 66127 - 24865\right) \overset{?}{\geq} 0 \rightarrow \text{true } \forall\, t \in$$

$$\left[2, \frac{66127}{24865}\right) \ \&\ \forall\, t \geq \frac{66127}{24865}, \text{LHS}_{(\bullet)} = \frac{(47358t - 65289)(4t-9)^2 + 53978t - 117383}{32} > 0$$

$$\therefore (1) \text{ is true } \therefore \sum_{cyc} \frac{1}{\alpha^3 + 3\alpha\beta\gamma + \beta^3} \geq \frac{18}{5\left(\sum_{cyc}\alpha^2\beta + \sum_{cyc}\alpha\beta^2\right)} \text{ and with } \alpha \equiv r_a,$$

$$\beta \equiv r_b, \gamma \equiv r_c, \text{we get : } \sum_{cyc} \frac{1}{r_a^3 + r_b^3 + 3s'^2r'} \geq \frac{9}{5s'^2(2R' - r')}, '' ='' \text{ iff } a = b = c$$