ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$\frac{3R}{r}-3\ge\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\ge\frac{R}{r}+1$$

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*Solution by Tapas Das-India*

$$\textit{Lemma}: \textit{In any } \Delta ABC \textit{ holds}:$$

$$(a+b+c)(a^2+b^2+c^2)\ge 3(a^2b+b^2c+c^2a)\ge$$
$$\ge(a+b+c)(ab+bc+ca)\ (\textit{Soumavo Chakraborty})$$

(Reference: Inequality in triangle by Dang Ngoc Minh -195 published at www.ssmrmh.ro )

Let s-a=x,s-b=y,s-c=z then a=y+z,b=x+z,c=x+y

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}=\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}=\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}$$

$$\frac{R}{r}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}=\frac{(x+y)(y+z)(z+x)}{4xyz}$$

$$\textit{We need to show}: \frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\ge\frac{R}{r}+1$$

$$or, \frac{y}{x}+\frac{z}{y}+\frac{x}{z}\ge\frac{(x+y)(y+z)(z+x)}{4xyz}+1$$

$$or, 4(x^2y+y^2z+z^2x)\ge\left(\sum x\right)\left(\sum xy\right)-xyz+4xyz$$

$$\left(3(x^2y+y^2z+z^2x)-\left(\sum x\right)\left(\sum xy\right)\right)+\left((x^2y+y^2z+z^2x)-3xyz\right)\ge 0\ \ \textit{true}$$

$$\left(\textit{using lemma}\ \&\ (x^2y+y^2z+z^2x)\overset{AM-GM}{\ge}3xyz\right)$$

$$\textit{Again we need to show}:$$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\le\frac{3R}{r}-3$$

$$\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\le 3\left(\frac{(x+y)(y+z)(z+x)}{4xyz}\right)-3$$

$$3(x+y)(y+z)(z+x)-12xyz\geq 4(x^2y+y^2z+z^2x)$$

$$3(x+y+z)(xy+yz+zx)-3xyz-12xyz\geq 4(x^2y+y^2z+z^2x)$$

$$3(x^2y+y^2z+z^2x)+3(x^2z+z^2y+y^2x)-6xyz\geq 4(x^2y+y^2z+z^2x)$$

$$3(x^2z+z^2y+y^2x)-6xyz\geq (x^2y+y^2z+z^2x)$$

$$(x+y+z)(xy+yz+zx)-6xyz\geq (x^2y+y^2z+z^2x)\ (using\ lemma)$$

$$(x^2y+y^2z+z^2x)+(x^2z+z^2y+y^2x)-3xyz\geq (x^2y+y^2z+z^2x)$$

$$(x^2z+z^2y+y^2x)-3xyz\geq 0\ true\ by\ AM-GM$$

**Equality holds for an equilateral triangle.**