ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{3+\sqrt[3]{4}}{2}abc+\frac{3-\sqrt[3]{4}}{6}(a^3+b^3+c^3)\le a^2b+b^2c+c^2a\le\frac{-9abc+2(a+b+c)^3}{15}$$

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Let $s-a=x, s-b=y, s-c=z$ and then : $s=x+y+z$ and $a=y+z$,

$$b=z+x, c=x+y \text{ and so, } 15\sum_{cyc}a^2b\le -9abc+2\left(\sum_{cyc}a\right)^3$$

$$\Leftrightarrow 16\left(\sum_{cyc}x\right)^3-9\prod_{cyc}(y+z)-15\sum_{cyc}\left((y+z)^2(z+x)\right)$$

$$\Leftrightarrow \sum_{cyc}x^3+9\sum_{cyc}xy^2-6\sum_{cyc}x^2y-12xyz\ \boxed{\overset{(*)}{\ge}}\ 0$$

Now, $F(x,y,z)=$ LHS of $(*)$ is a cyclic homogeneous polynomial of degree 3 and so, via CD3 (cyclic polynomial of degree 3) theorem, in order to prove $(*)$, it suffices to prove : $F(1,1,1)\ge 0$ AND $F(x,y,0)\ge 0$ and indeed, $F(1,1,1)=0$ and $F(x,y,0)$ $=x^3+y^3+9xy^2-6x^2y=x(x^2-6xy+9y^2)+y^3=x(x-3y)^2+y^3>0$ $(\because x,y>0)$ $\therefore$ via CD3 theorem, $F(x,y,z)=$ LHS of $(*)\ge 0$

$$\therefore a^2b+b^2c+c^2a\le\frac{-9abc+2(a+b+c)^3}{15} \text{ and again,}$$

$$\frac{3+\sqrt[3]{4}}{2}.abc+\frac{3-\sqrt[3]{4}}{2}.(a^3+b^3+c^3)\le a^2b+b^2c+c^2a$$

$$\Leftrightarrow \sqrt[3]{4}.\left(\sum_{cyc}a^3-3abc\right)+6\sum_{cyc}a^2b-3\left(\sum_{cyc}a^3+3abc\right)\ge 0$$

$$\Leftrightarrow m\left(\sum_{cyc}(y+z)^3-3\prod_{cyc}(y+z)\right)+6\sum_{cyc}\left((y+z)^2(z+x)\right)-$$

$$3\left(\sum_{cyc}(y+z)^3+3\prod_{cyc}(y+z)\right)\ (m=\sqrt[3]{4})$$

$$\Leftrightarrow m\sum_{cyc}x^3+(9-3m)xyz-3\sum_{cyc}x^2y\ \boxed{\overset{(**)}{\ge}}\ 0$$

Now, $G(x,y,z)=$ LHS of $(**)$ is a cyclic homogeneous polynomial of degree 3 and so, via CD3 (cyclic polynomial of degree 3) theorem, in order to prove $(**)$, it suffices to prove : $G(1,1,1)\ge 0$ AND $G(x,y,0)\ge 0$ and indeed, $G(1,1,1)$

$$= 3m + 9 - 3m - 9 = 0 \text{ and } G(x, y, 0) = \sqrt[3]{4}.(x^3 + y^3) - 3x^2y \overset{?}{\geq} 0$$

$$\Leftrightarrow \sqrt[3]{4}.(t^3 + 1) - 3t^2 \overset{?}{\geq} 0 \left(t = \frac{x}{y}\right) \Leftrightarrow 4(t^3 + 1)^3 - 27t^6 \overset{?}{\geq} 0$$

$$\Leftrightarrow 4t^9 - 15t^6 + 12t^3 + 4 \overset{?}{\geq} 0 \Leftrightarrow (t^3 - 2)^2(4t^3 + 1) \overset{?}{\geq} 0 \to \text{true} \because t > 0$$

$$\therefore \text{ via CD3 theorem, } G(x, y, z) = \text{LHS of } (**) \geq 0$$

$$\therefore \frac{3 + \sqrt[3]{4}}{2}.abc + \frac{3 - \sqrt[3]{4}}{6}.(a^3 + b^3 + c^3) \leq a^2b + b^2c + c^2a \text{ and so,}$$

$$\frac{3 + \sqrt[3]{4}}{2}.abc + \frac{3 - \sqrt[3]{4}}{6}.(a^3 + b^3 + c^3) \leq a^2b + b^2c + c^2a \leq$$

$$\frac{-9abc + 2(a + b + c)^3}{15} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$