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In ΔABC the following relationship holds:

$$\sqrt{\frac{r_b}{r_c}}+\sqrt{\frac{r_c}{r_b}}\le\frac{2s}{3\sqrt{3}r}$$

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$$r_a=\frac{F}{s-a},r_b=\frac{F}{s-b},r_c=\frac{F}{s-c}$$

$$\textit{Using above result we get}:$$

$$\sqrt{\frac{r_b}{r_c}}+\sqrt{\frac{r_c}{r_b}}=\sqrt{\frac{s-c}{s-b}}+\sqrt{\frac{s-b}{s-c}}=\frac{s-c+s-b}{\sqrt{(s-b)(s-c)}}=\frac{2s-(b+c)}{\sqrt{(s-b)(s-c)}}=$$

$$=\frac{a\sqrt{s(s-a)}}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{a\sqrt{s(s-a)}}{F}=\frac{a\sqrt{s(s-a)}}{rs}$$

$$\textit{We need to show}:$$

$$\frac{a\sqrt{s(s-a)}}{rs}\le\frac{2s}{3\sqrt{3}r}\ or\ a\sqrt{s-a}\le\frac{2s\sqrt{s}}{3\sqrt{3}r}$$

$$or\ \ a\sqrt{s-a}\le 2\left(\frac{s}{3}\right)^{\frac{3}{2}}\ or\ a^2(s-a)\overset{squaring}{\le}\frac{4s^3}{27}$$

$$4s^3-27a^2s-27a^3\ge 0\ or\ 4x^3-27x+27\overset{\frac{s}{a}=x>1}{\ge}0$$

$$(2x-3)^2(x+3)\ge 0$$

Equality holds for an equilateral triangle.