ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC following relationship holds :

$$\left(\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)\left(\frac{r_a}{n_a}+\frac{r_b}{n_b}+\frac{r_c}{n_c}\right)\ge 9$$

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$$\text{Firstly,}\sum_{cyc} a n_a \overset{\text{CBS}}{\le} \sqrt{\sum_{cyc} a}\,.\sqrt{\sum_{cyc} a n_a^2} \overset{\text{Bogdan Fustei}}{=} \sqrt{2s}.\sqrt{\sum_{cyc} a(s^2-2h_a r_a)}$$

$$=\sqrt{2s}.\sqrt{s^2(2s)-4rs(4R+r)} \therefore \sum_{cyc} a n_a \overset{①}{\le} 2s.\sqrt{s^2-8Rr-2r^2} \text{ and}$$

$$\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{2(s-a)}{a}=\frac{s(s^2+4Rr+r^2)}{2Rrs}-6 \therefore \sum_{cyc}\frac{h_a}{r_a}\overset{②}{=}\frac{s^2-8Rr+r^2}{2Rr}$$

$$\text{Now,}\sum_{cyc}\frac{r_a}{n_a}=\sum_{cyc}\frac{2h_a r_a}{2n_a h_a}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{s^2-n_a^2}{2n_a h_a}=\frac{s^2}{4rs}.\sum_{cyc}\frac{a}{n_a}-\frac{1}{4rs}.\sum_{cyc} a n_a$$

$$=\frac{s}{4r}.\sum_{cyc}\frac{a^2}{a n_a}-\frac{1}{4rs}.\sum_{cyc} a n_a \overset{\text{Bergstrom}}{\ge} \frac{s}{4r}.\frac{4s^2}{\sum_{cyc} a n_a}-\frac{1}{4rs}.\sum_{cyc} a n_a \overset{\text{via ①}}{\ge}$$

$$\frac{s}{4r}.\frac{4s^2}{2s.\sqrt{s^2-8Rr-2r^2}}-\frac{1}{4rs}.2s.\sqrt{s^2-8Rr-2r^2}=\frac{s^2-(s^2-8Rr-2r^2)}{2r.\sqrt{s^2-8Rr-2r^2}}$$

$$\therefore \left(\sum_{cyc}\frac{r_a}{n_a}\right)^2 \ge \frac{(4R+r)^2}{s^2-8Rr-2r^2} \overset{\text{via ②}}{\Rightarrow} \left(\sum_{cyc}\frac{h_a}{r_a}\right)^2.\left(\sum_{cyc}\frac{r_a}{n_a}\right)^2 \ge$$

$$\frac{(s^2-8Rr+r^2)^2}{4R^2r^2}.\frac{(4R+r)^2}{s^2-8Rr-2r^2}\overset{?}{\ge} 81 \Leftrightarrow$$

$$(4R+r)^2(s^2-8Rr+r^2)^2-324R^2r^2(s^2-8Rr-2r^2)\ \boxed{\overset{?}{\ge}\ (*)}\ 0 \text{ and}$$

$$\because P=(4R+r)^2(s^2-16Rr+5r^2)^2+$$

$$4r(64R^3-81R^2r-12Rr^2-2r^3)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\ge} P$$

$$\Leftrightarrow 256t^4-776t^3+519t^2+16t+4\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(256t^2+248t+487)+972\Big)\overset{?}{\ge}0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (*) \text{ is true} \therefore \left(\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\right)\left(\frac{r_a}{n_a}+\frac{r_b}{n_b}+\frac{r_c}{n_c}\right)\ge 9\ \forall\ \Delta\ ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)