ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y, s-c=z$ and then : $x,y,z>0$ and $\sum_{cyc} a = 2\sum_{cyc} x$ and consequently, $a=y+z, b=z+x, c=x+y$ and WLOG we may assume $a=\min\{a,b,c\}$ and then 2 cases arise :

$\boxed{\text{Case 1}}$ $a \leq b \leq c \Rightarrow x \geq y \geq z$ and then : $\max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}$

$$\Leftrightarrow w_a \geq m_c \Leftrightarrow \frac{4bc}{(b+c)^2}.s(s-a) \geq \frac{2a^2+2b^2-c^2}{4}$$

$$\Leftrightarrow 16(z+x)(x+y).(x+y+z)x \geq$$

$$(2x+y+z)^2(2(y+z)^2+2(z+x)^2-(x+y)^2) \Leftrightarrow$$

$$12x^4+12x^3z+36x^3y-17x^2z^2+22x^2yz+19x^2y^2-20xz^3-22xyz^2-4xy^2z$$
$$-2xy^3-4z^4-12yz^3-13y^2z^2-6y^3z-y^4 \geq 0$$

$$\Leftrightarrow (12x+12y+4z)(x^3-z^3)+22xyz(x-z)+(8xz+13y^2)(x^2-z^2)+$$
$$y(x^3-y^3)+2xy(x^2-y^2)+11x^2(xy-z^2)+6x^2(y^2-z^2)+4xy(x^2-yz)+$$
$$6y(x^3-y^2z) \geq 0 \rightarrow \text{true} \because x \geq y \geq z > 0 \therefore \max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}$$

$\boxed{\text{Case 2}}$ $a \leq c \leq b \Rightarrow x \geq z \geq y$ and then : $\max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}$

$$\Leftrightarrow w_a \geq m_b \Leftrightarrow \frac{4bc}{(b+c)^2}.s(s-a) \geq \frac{2a^2+2c^2-b^2}{4}$$

$$\Leftrightarrow 16(z+x)(x+y).(x+y+z)x \geq$$

$$(2x+y+z)^2(2(y+z)^2+2(x+y)^2-(z+x)^2) \Leftrightarrow$$

$$12x^4+12x^3y+36x^3z-17x^2y^2+22x^2yz+19x^2z^2-20xy^3-22xy^2z-4xyz^2$$
$$-2xz^3-4y^4-12y^3z-13y^2z^2-6yz^3-z^4 \geq 0$$

$$\Leftrightarrow (12x+12z+4y)(x^3-y^3)+22xyz(x-y)+(8xy+13z^2)(x^2-y^2)+$$
$$z(x^3-z^3)+2xz(x^2-z^2)+11x^2(xz-y^2)+6x^2(z^2-y^2)+4xz(x^2-yz)+$$
$$6z(x^3-yz^2) \geq 0 \rightarrow \text{true} \because x \geq z \geq y > 0 \therefore \max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}$$

$\therefore$ combining both cases and 2 other analogs, we conclude that :

$$\max\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\} \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$