ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a = \min\{a, b, c\}$ the following relationship ho$l$ds :

$$\frac{b}{c} + \frac{c}{b} \le 1 + \frac{R}{2r}$$

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$$\text{Let } s - a = x, s - b = y, s - c = z \text{ and then : } x, y, z > 0 \text{ and}$$

$$\sum_{cyc} a = 2\sum_{cyc} x \text{ and consequently, } a = y + z, b = z + x, c = x + y \text{ and then :}$$

$$\frac{b}{c} + \frac{c}{b} \le 1 + \frac{R}{r} \Leftrightarrow \frac{z + x}{x + y} + \frac{x + y}{z + x} \le 1 + \frac{(x + y)(y + z)(z + x)}{8xyz}$$

$$\Leftrightarrow (x + y)^2(y + z)(z + x)^2 \ge 8xyz((z + x)^2 + (x + y)^2)$$

$$\Leftrightarrow x^4y + x^4z + y^3z^2 + y^2z^3 - 2x^2y^2z - 2x^2yz^2 + 2x^3(y - z)^2 +$$

$$x^2\left(y^3 + z^3 - yz(y + z)\right) - 6xyz(y - z)^2 \ge 0$$

$$\Leftrightarrow (x^4y + y^3z^2 - 2x^2y^2z) + (x^4z + y^2z^3 - 2x^2yz^2) +$$

$$x(y - z)^2(2x^2 + xy + xz - 3yz) \overset{(*)}{\ge} 0$$

$$\text{Now, } a = \min\{a, b, c\} \Rightarrow y + z \le z + x, x + y \Rightarrow x \ge y, z \Rightarrow x^2 \ge yz,$$

$$xy \ge yz, xz \ge yz \Rightarrow 2x^2 + xy + xz \overset{(1)}{\ge} 3yz \text{ and also, } x^4y + y^3z^2 \overset{AM-GM}{\ge} 2x^2y^2z$$

$$\Rightarrow x^4y + y^3z^2 - 2x^2y^2z \overset{(2)}{\ge} 0 \text{ and } x^4z + y^2z^3 \overset{AM-GM}{\ge} 2x^2yz^2$$

$$\Rightarrow x^4z + y^2z^3 - 2x^2yz^2 \overset{(3)}{\ge} 0 \therefore (1), (2) \text{ and } (3) \Rightarrow (*) \text{ is true} \therefore \frac{b}{c} + \frac{c}{b} \le 1 + \frac{R}{2r}$$

$$\forall \Delta ABC \text{ with } a = \min\{a, b, c\}, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$