ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{3}(a+b+c)(a^2+b^2+c^2) \ge a^2b+b^2c+c^2a \ge \frac{1}{3}(a+b+c)(ab+bc+ca)$$

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$$\text{Let } s-a=x, s-b=y, s-c=z \text{ and then : } \sum_{cyc} a = 2\sum_{cyc} x \text{ and}$$

$$\text{consequently, } a=y+z, b=z+x, c=x+y \text{ and then : LH inequality} \Leftrightarrow$$

$$2\left(\sum_{cyc} x\right)\left(\sum_{cyc}(y+z)^2\right) \ge 3\sum_{cyc}\left((z+x)(y+z)^2\right) \Leftrightarrow \sum_{cyc} x^3 + 2\sum_{cyc} x^2y \ge$$

$$\sum_{cyc} xy^2 + 6xyz \to \text{true} \because \sum_{cyc} x^3 + 2\sum_{cyc} x^2y = \sum_{cyc} x^2y + \left(\sum_{cyc} x^3 + \sum_{cyc} z^2x\right)$$

$$\overset{AM-GM}{\ge} 3xyz + 2\sum_{cyc} zx^2 \overset{AM-GM}{\ge} 3xyz + 3xyz + \sum_{cyc} xy^2$$

$$\therefore \frac{1}{3}(a+b+c)(a^2+b^2+c^2) \ge a^2b+b^2c+c^2a \text{ and also, RH inequality}$$

$$\Leftrightarrow 3\sum_{cyc}\left((z+x)(y+z)^2\right) \ge 2\left(\sum_{cyc} x\right)\left(\sum_{cyc}\left((y+z)(z+x)\right)\right)$$

$$\Leftrightarrow \sum_{cyc} x^3 + \sum_{cyc} xy^2 \ge 2\sum_{cyc} x^2y \Leftrightarrow \sum_{cyc} x(x-y)^2 \ge 0 \to \text{true}$$

$$\therefore a^2b+b^2c+c^2a \ge \frac{1}{3}(a+b+c)(ab+bc+ca) \text{ \& so,}$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \ge \sum_{cyc} a^2b \ge \frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \forall \Delta \text{ ABC,}$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$