ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{a^2}{a+2r}+\frac{b^2}{b+2r}+\frac{c^2}{c+2r}\geq 3s-(m_a+m_b+m_c)$$

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$$\frac{a^2}{a+2r}+\frac{b^2}{b+2r}+\frac{c^2}{c+2r}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a+6r}=\frac{2s^2}{s+3r}\overset{?}{\geq}3s-\sum_{cyc}m_a$$

$$\Leftrightarrow \frac{-s^2-9rs}{s+3r}\overset{?}{\geq}-\sum_{cyc}m_a\Leftrightarrow\sum_{cyc}m_a\overset{?}{\geq}\frac{s^2+9rs}{s+3r}\to ① \text{ and when : }\sum_{cyc}m_a\geq\sqrt{3}s,$$

then, in order to prove ①, it suffices to prove : $\sqrt{3}s+3\sqrt{3}r\overset{?}{\geq}s+9r$

$\Leftrightarrow(\sqrt{3}-1)s\overset{?}{\geq}3\sqrt{3}(\sqrt{3}-1)r\to$ true via Mitrinovic $\Rightarrow$ ① is true and we now focus on the scenario when : $\sum_{cyc}m_a\leq\sqrt{3}s$ and we shall now prove :

$$\frac{2\left(\sum_{cyc}m_a\right)^3}{3\left(\sum_{cyc}m_a\right)^2+27F}\overset{(*)}{\geq}\sum_{cyc}m_a-\frac{1}{2}\sum_{cyc}a$$

$$\Leftrightarrow 2\left(\sum_{cyc}m_a\right)^3+27rs^2\geq 3\left(\sum_{cyc}m_a\right)^3-3s\left(\sum_{cyc}m_a\right)^2+27F\left(\sum_{cyc}m_a\right) \text{ and}$$

$$\because\sum_{cyc}m_a\leq\sqrt{3}s\Rightarrow 27rs^2\geq 9r\left(\sum_{cyc}m_a\right)^2 \therefore \text{ it suffices to prove :}$$

$$9rt+3st-27rs-t^2\overset{?}{\geq}0\left(t=\sum_{cyc}m_a\right)\Leftrightarrow t^2-(9r+3s)t+27rs\overset{?}{\leq}0$$

$$\Leftrightarrow t\overset{?}{\leq}\frac{9r+3s+\sqrt{(9r+3s)^2-108rs}}{2}\wedge t\overset{?}{\geq}\frac{9r+3s-\sqrt{(9r+3s)^2-108rs}}{2}$$

$$\Leftrightarrow t\overset{?}{\leq}\frac{9r+3s+3s-9r}{2}\wedge t\overset{?}{\geq}\frac{9r+3s-3s+9r}{2}\Leftrightarrow t\overset{?}{\leq}3s\wedge t\overset{?}{\geq}9r\to\text{true}$$

$$\because t=\sum_{cyc}m_a\leq\sqrt{3}s<3s\wedge t\geq\sum_{cyc}h_a\geq 9r\Rightarrow(*)\text{ is true and implementing }(*)$$

on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose sides and area as a consequence of trivial calculations are : $\frac{a}{2},\frac{b}{2},\frac{c}{2}$ and $\frac{F}{3}$ respectively, we get :

$$\frac{2\left(\sum_{cyc}\frac{a}{2}\right)^3}{3\left(\sum_{cyc}\frac{a}{2}\right)^2+9F} \geq \sum_{cyc}\frac{a}{2}-\frac{1}{2}\sum_{cyc}\frac{2m_a}{3} \Rightarrow \frac{2s^2}{3s+9r} \geq s-\frac{1}{3}\sum_{cyc}m_a$$

$$\Rightarrow \frac{-s^2-9rs}{s+3r} \geq \sum_{cyc}m_a \Rightarrow \sum_{cyc}m_a \geq \frac{s^2+9rs}{s+3r} \Rightarrow (1) \text{ is true and}$$

combining both cases, (1) is true $\forall \Delta ABC \Rightarrow \frac{a^2}{a+2r}+\frac{b^2}{b+2r}+\frac{c^2}{c+2r} \geq 3s-(m_a+m_b+m_c)$ $\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)