ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $following$ $relationship$ $holds$:

$$n_a \le R + \sqrt{R^2 - rr_a}$$

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$$4R(R-2r) \overset{?}{\ge} (b-c)^2$$

$$\Leftrightarrow 4R^2\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right) \overset{?}{\ge} 16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$

$$\Leftrightarrow 1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2} \overset{?}{\ge} 4\sin^2\frac{A}{2} - 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2} \overset{?}{\ge} 0$$

$$\Leftrightarrow \left(1 - 2\sin\frac{A}{2}\cos\frac{B-C}{2}\right)^2 \overset{?}{\ge} 0 \to \text{true} \therefore 4R(R-2r) \ge (b-c)^2 \to (1)$$

If $n_a \le R$, then, proposed $inequality$ is $trivially$ true and when $n_a > R$, then :

$$n_a \le R + \sqrt{R^2 - rr_a} \Leftrightarrow (n_a - R)^2 \le R^2 - rr_a \Leftrightarrow n_a^2 + \frac{a^2 - (b-c)^2}{4} \le 2Rn_a$$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2 + \frac{a^2 - (b-c)^2}{4} \le 2Rn_a$$

$$\Leftrightarrow \frac{(b+c)^2 - a^2}{4} + \frac{a^2 - (b-c)^2}{4} + \frac{s}{a}(b-c)^2 \le 2Rn_a \Leftrightarrow \frac{bc}{2R} + \frac{s}{a}.\frac{(b-c)^2}{2R} \le n_a$$

$$\Leftrightarrow \frac{n_a}{h_a} \ge 1 + \frac{s}{a}.\frac{(b-c)^2}{2R}.\frac{2R}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \ge \left(\frac{s(b-c)^2 + abc}{abc}\right)^2 - 1 \overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$

$$4R^2.\frac{s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)}{a^2}(b-c)^2}{b^2c^2} \ge$$

$$\frac{s^2(b-c)^4 + 2sabc(b-c)^2}{a^2b^2c^2} \Leftrightarrow 4R^2s^2(b-c)^2 \ge s^2(b-c)^4 + 8Rrs^2.(b-c)^2$$

$$\Leftrightarrow s^2(b-c)^2(4R^2 - 8Rr - (b-c)^2) \ge 0 \to \text{true via } (1)$$

$$\therefore n_a \le R + \sqrt{R^2 - rr_a}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \left(\begin{matrix}\sin\frac{A}{2}.\cos\frac{B-C}{2} = \frac{1}{2} \text{ with } \Delta\ ABC \\ \text{being equilateral, } a \text{ special case}\end{matrix}\right)$$

$$\text{or } (b = c \ne a) \text{ (QED)}$$