ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$w_a + 3h_a \le 4g_a$$

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$$w_a + 3h_a \overset{CBS}{\le} \sqrt{1+1+1+1}.\sqrt{w_a^2 + 3h_a^2} \overset{?}{\le} 4g_a \Leftrightarrow w_a^2 + 3h_a^2 \overset{?}{\le} 4g_a^2$$

$$\Leftrightarrow s(s-a) - \frac{s(s-a)}{(2s-a)^2}(b-c)^2 + 3s(s-a) - \frac{3s(s-a)}{a^2}(b-c)^2 \overset{?}{\le}$$

$$4s(s-a) - \frac{4(s-a)}{a}(b-c)^2 \Leftrightarrow \left(\frac{s}{(2s-a)^2} + \frac{3s}{a^2} - \frac{4}{a}\right)(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{3s(2s-a)^2 + sa^2 - 4a(2s-a)^2}{a^2(2s-a)^2}.(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{(s-a)^2(3s-a)}{a^2(2s-a)^2}.(b-c)^2 \overset{?}{\ge} 0 \to \text{true} \because s > a \therefore w_a + 3h_a \le 4g_a \ \forall \Delta \text{ ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is isosceles with } b = c \text{ (QED)}$$