ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$ab+bc+ca \geq 4\sqrt{3}F\cdot\frac{m_a+m_b+m_c}{h_a+h_b+h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\left(\sum_{cyc}ab\right)^2\left(\sum_{cyc}h_a\right)^2}{\left(\sum_{cyc}m_a\right)^2.48r^2s^2} \overset{\text{Chu-Yang}}{\geq} \frac{(s^2+4Rr+r^2)^4}{192R^2r^2s^2(4s^2-16Rr+5r^2)} \overset{?}{\geq} 1$$

$$\Leftrightarrow s^8+(16Rr+4r^2)s^6-r^2(672R^2-48Rr-6r^2)s^4+$$

$$r^3(3328R^3-768R^2r+48Rr^2+4r^3)s^2+r^4(4R+r)^4 \boxed{\overset{(*)}{\geq}} 0 \text{ and } \because P=$$

$$(s^2-16Rr+5r^2)^4+16(5Rr-r^2)(s^2-16Rr+5r^2)^3+$$

$$32r^2(51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2+$$

$$64r^3(164R^3-195R^2r+60Rr^2-4r^3)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow 196t^4-560t^3+375t^2-80t+4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(196t^3-168t^2+39t-2) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore ab+bc+ca \geq 4\sqrt{3}F.\frac{m_a+m_b+m_c}{h_a+h_b+h_c}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$