ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{9abc}{a+b+c}\ge 4\sqrt{3}F\cdot\frac{w_a+w_b+w_c}{h_a+h_b+h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\left(\frac{9abc}{a+b+c}\right)^2.\left(\sum_{cyc}h_a\right)^2}{48r^2s^2.\left(\sum_{cyc}w_a\right)^2}\overset{\text{Dang Ngoc Minh}}{\ge}\frac{324R^2r^2.(s^2+4Rr+r^2)^2}{192R^2r^2s^2.(s^2+21Rr+12r^2)}=$$

$$\frac{27(s^2+4Rr+r^2)^2}{16s^2(s^2+21Rr+12r^2)}\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 90;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$\overset{?}{\ge}1\Leftrightarrow 11s^4-(120Rr+138r^2)s^2+27r^2(4R+r)^2\ \boxed{\overset{(*)}{\ge}}\ 0\text{ and }\because P=$$

$$11(s^2-16Rr+5r^2)^2+8(29Rr-31r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\ge}0$$

$$\therefore\text{ in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\ge}P$$

$$\Leftrightarrow 16r^2(R-2r)(83R-31r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{9abc}{a+b+c}\ge 4\sqrt{3}F.\frac{w_a+w_b+w_c}{h_a+h_b+h_c}\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$