ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$a^2+b^2+c^2 \ge 4\sqrt{3}F\cdot\frac{r_a+r_b+r_c}{w_a+w_b+w_c}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}w_a\right)^2 \ge 4(s^2-4Rr-r^2)^2(s^2+12Rr+30r^2) \overset{?}{\ge}$$

$$48r^2s^2(4R+r)^2 \left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 90;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\Leftrightarrow s^6+(4Rr+28r^2)s^4-r^2(272R^2+352Rr+71r^2)s^2+$$

$$r^3(192R^3+576R^2r+252Rr^2+30r^3) \boxed{\overset{(\bullet)}{\ge}} 0 \text{ and } \because P=(s^2-16Rr+5r^2)^3+$$

$$(52Rr+13r^2)(s^2-16Rr+5r^2)^2+12r^2(52R^2+2Rr-23r^2)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow 20t^3-21t^2-48t+20 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(20t^2+19t-10) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true} \therefore a^2+b^2+c^2 \ge 4\sqrt{3}F.\frac{r_a+r_b+r_c}{w_a+w_b+w_c}\ \forall\,\Delta\,\text{ABC},$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$