ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{R}{r} \geq 1 + \frac{a^4 + b^4 + c^4}{abc(a+b+c)} \geq 2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{R}{r} \geq 1 + \frac{a^4 + b^4 + c^4}{abc(a+b+c)}$$

$$\Leftrightarrow \frac{R}{r} \geq \frac{8Rrs^2 + 2(s^2 + 4Rr + r^2)^2 - 32Rrs^2 - 16r^2s^2}{8Rrs^2}$$

$$\Leftrightarrow -s^4 + (4R^2 + 4Rr + 6r^2)s^2 - r^2(4R+r)^2 \boxed{\overset{(\bullet)}{\geq}} 0 \text{ and } \because P =$$

$$-(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3) \overset{\text{Double-Rouche}}{\geq} 0 \therefore \text{ in order}$$

$$\text{to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow 4r(R(4R+r)^2 - (4R-2r)s^2) \overset{?}{\geq} 0 \to \text{true via Blundon} - \text{Gerretsen}$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{R}{r} \geq 1 + \frac{a^4 + b^4 + c^4}{abc(a+b+c)} \text{ and also}, 1 + \frac{a^4 + b^4 + c^4}{abc(a+b+c)} \geq 2$$

$$\Leftrightarrow \sum_{cyc} a^4 \geq abc \sum_{cyc} a \to \text{true} \because \sum_{cyc} a^4 \geq \sum_{cyc} a^2b^2 \geq abc \sum_{cyc} a$$

$$\therefore \frac{R}{r} \geq 1 + \frac{a^4 + b^4 + c^4}{abc(a+b+c)} \geq 2 \ \forall \ \Delta \ ABC, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$