ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2(ab+bc+ca)-(a^2+b^2+c^2)\geq 4F.\sqrt{3+\frac{16R^2(R-r)(R-2r)}{16R^4-16R^3r-4R^2r^2-r^4}}$$

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$$\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4F}=\frac{4R+r}{s}\overset{\text{Rouche}}{\geq}$$

$$\frac{4R+r}{\sqrt{2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}}}\overset{?}{\geq}\sqrt{3+\frac{16R^2(R-r)(R-2r)}{16R^4-16R^3r-4R^2r^2-r^4}}$$

$$\Leftrightarrow (4R+r)^2(16R^4-16R^3r-4R^2r^2-r^4)\overset{?}{\geq}$$

$$\left(3(16R^4-16R^3r-4R^2r^2-r^4)+16R^2(R-r)(R-2r)\right)\begin{pmatrix}2R^2+10Rr-r^2+\\2(R-2r).\sqrt{R^2-2Rr}\end{pmatrix}$$

$$\Leftrightarrow 2(R-2r)(64R^5-160R^4r+84R^3r^2-4R^2r^3-5Rr^4+r^5)\underset{(*)}{\overset{?}{\geq}}$$

$$2(R-2r).\sqrt{R^2-2Rr}.(64R^4-96R^3r+20R^2r^2-3r^4)\ \textit{and}\ \because$$

$$64R^5-160R^4r+84R^3r^2-4R^2r^3-5Rr^4+r^5$$

$$=(R-2r)(64R^4-32R^3r+20R^2r^2+36Rr^3+67r^4)+135r^5\overset{\text{Euler}}{\geq}135r^5>0$$

$$\textit{and}\because R-2r\overset{\text{Euler}}{\geq}0\ \therefore\ \text{in order to prove } (*),\text{it suffices to prove :}$$

$$(64R^5-160R^4r+84R^3r^2-4R^2r^3-5Rr^4+r^5)^2\overset{?}{>}$$

$$(R^2-2Rr)(64R^4-96R^3r+20R^2r^2-3r^4)^2$$

$$\Leftrightarrow r^5(512R^5+128R^4r-32R^3r^2+8R^2r^3+8Rr^4+r^5)\overset{?}{>}0\rightarrow \text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow (*)\text{ is true}\ \therefore\ 2(ab+bc+ca)-(a^2+b^2+c^2)\geq$$

$$4F.\sqrt{3+\frac{16R^2(R-r)(R-2r)}{16R^4-16R^3r-4R^2r^2-r^4}}\ \forall\,\Delta\,\text{ABC},''=''\ \text{iff}\ \Delta\,\text{ABC is equilateral (QED)}$$