ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$4w_bw_c \le a\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

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$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \overset{\text{GM-HM}}{\ge} \sum_{\text{cyc}} \frac{2bc}{b+c} = \frac{2\sum_{\text{cyc}}\left(bc(c+a)(a+b)\right)}{(a+b)(b+c)(c+a)}$$

$$= \frac{2\sum_{\text{cyc}}\left(bc\left(a^2+\sum_{\text{cyc}}ab\right)\right)}{(a+b)(b+c)(c+a)} = \frac{2\left(abc\sum_{\text{cyc}}a+\left(\sum_{\text{cyc}}ab\right)^2\right)}{(a+b)(b+c)(c+a)}$$

$$\Rightarrow a\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \ge \frac{2a(8Rrs^2+(s^2+4Rr+r^2)^2)}{(a+b)(b+c)(c+a)} \to (1)$$

$$\text{Now, } 4w_bw_c = 4.\frac{2ca\cos\frac{B}{2}.2ab\cos\frac{C}{2}.\left(4R\cos\frac{A}{2}\cos\frac{B-C}{2}\right)}{(a+b)(b+c)(c+a)} \overset{0<\cos\frac{B-C}{2}\le 1}{\le}$$

$$\frac{64R.\frac{s}{4R}.4Rrs.a}{(a+b)(b+c)(c+a)} \overset{?}{\le} \frac{2a(8Rrs^2+(s^2+4Rr+r^2)^2)}{(a+b)(b+c)(c+a)}$$

$$\Leftrightarrow 8Rrs^2+(s^2+4Rr+r^2)^2 \overset{?}{\ge} 32Rrs^2$$

$$\Leftrightarrow s^4-(16Rr-2r^2)s^2+r^2(4R+r)^2 \overset{?}{\ge} 0 \text{ and } \because s^4 \overset{\text{Gerretsen}}{\ge} (16Rr-5r^2)s^2$$

$$\therefore \text{ it suffices to prove : } (16Rr-5r^2)s^2-(16Rr-2r^2)s^2+r^2(4R+r)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow r^2((4R+r)^2-3s^2) \overset{?}{\ge} 0 \to \text{true via Doucet (or Trucht)}$$

$$\Rightarrow 4w_bw_c \le \frac{2a(8Rrs^2+(s^2+4Rr+r^2)^2)}{(a+b)(b+c)(c+a)} \overset{\text{via (1)}}{\le} a\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$\forall\, \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)