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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{n_a}{r_a}\le 12+\frac{1}{r}\sum_{cyc}(n_a-2h_a)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$12+\frac{1}{r}\sum_{cyc}(n_a-2h_a)-\sum_{cyc}\frac{n_a}{r_a}=12-\sum_{cyc}\frac{4sbc}{4Rrs}+\sum_{cyc}\left(n_a.\left(\frac{s}{rs}-\frac{s-a}{rs}\right)\right)$$

$$=2\left(6-\sum_{cyc}\frac{bc(a+b+c)}{abc}+\sum_{cyc}\frac{n_a}{h_a}\right)\overset{\text{Ben Ajiba}}{\ge}$$

$$2\left(\frac{1}{abc}\left(6abc-3abc-\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)+\sum_{cyc}\frac{b^2-bc+c^2}{bc}\right)$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 28;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)=$$

$$2\left(\frac{1}{abc}\left(6abc-3abc-\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)+\frac{1}{abc}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2-3abc\right)\right)$$

$$=0\therefore\sum_{cyc}\frac{n_a}{r_a}\le 12+\frac{1}{r}\sum_{cyc}(n_a-2h_a)\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$