ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{n_a-|b-c|}{n_a+p_a}\left(1+\sqrt{\frac{g_a}{w_a}}\right)\right)>1$$

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$$m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow} \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)$$

$$\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2+\frac{(b-c)^4}{324}+$$

$$\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\to\text{true (strict inequality)}\therefore m_a n_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_a n_a\geq p_a^2\to(1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a w_a\to(2)\therefore(1)\bullet(2)\Rightarrow(m_a n_a)(n_a g_a)\geq p_a^2.m_a w_a$$

$$\Rightarrow\frac{n_a}{p_a}\geq\sqrt{\frac{w_a}{g_a}}\Rightarrow 1+\sqrt{\frac{g_a}{w_a}}\geq 1+\frac{p_a}{n_a}\Rightarrow\frac{1+\sqrt{\frac{g_a}{w_a}}}{n_a+p_a}\geq\frac{1}{n_a}\to (1)\text{ and also,}$$

$$n_a^2\overset{\text{Bogdan Fustei}}{=}s(s-a)+\frac{s(b-c)^2}{a}>\frac{s(b-c)^2}{a}>(b-c)^2\Rightarrow n_a-|b-c|>0$$

$$\textit{and so,}\ \frac{n_a - |b-c|}{n_a + p_a}\left(1 + \sqrt{\frac{g_a}{w_a}}\right) \overset{\text{via } ①}{\geq} \frac{n_a - |b-c|}{n_a}\ \textit{and analogs}$$

$$\Rightarrow \sum_{\text{cyc}}\left(\frac{n_a - |b-c|}{n_a + p_a}\left(1 + \sqrt{\frac{g_a}{w_a}}\right)\right) \geq \sum_{\text{cyc}}\frac{n_a - |b-c|}{n_a} = 3 - \sum_{\text{cyc}}\frac{|b-c|}{n_a} \overset{?}{>} 1$$

$$\Leftrightarrow \sum_{\text{cyc}}\frac{|b-c|}{n_a} \boxed{\underbrace{\overset{?}{<}}_{(*)}}\ 2\ \textit{and now,}\ \frac{|b-c|}{n_a} \overset{?}{<} \frac{a}{s} \Leftrightarrow a^2 n_a^2 \overset{?}{>} s^2(b-c)^2 \overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$

$$a^2\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{>} s^2(b-c)^2 \Leftrightarrow a^2 s(s-a) \overset{?}{>} (s^2 - sa)(b-c)^2$$

$$\Leftrightarrow s(s-a)(a^2 - (b-c)^2) \overset{?}{>} 0 \rightarrow \text{true} \therefore \frac{|b-c|}{n_a} < \frac{a}{s}\ \textit{and analogs}$$

$$\Rightarrow \sum_{\text{cyc}}\frac{|b-c|}{n_a} < \frac{1}{s}.\sum_{\text{cyc}} a = \frac{2s}{s} = 2 \Rightarrow (*)\ \text{is true}$$

$$\therefore \sum_{\text{cyc}}\left(\frac{n_a - |b-c|}{n_a + p_a}\left(1 + \sqrt{\frac{g_a}{w_a}}\right)\right) > 1\ \forall\ \Delta\, ABC\ (QED)$$