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In any Δ ABC the following relationships hold :

$$2.\ \sqrt{\frac{h_a\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}{n_a}}+\frac{r_a+r}{\sqrt{2R}}\le 2\sqrt{2R}\ \text{and}$$

$$\sqrt{2R\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\le\left(\frac{r_b+r_c}{2}-r\right).\sqrt{\frac{n_a}{h_a}}$$

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$$2\sqrt{\frac{h_a\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}{n_a}}+\frac{r_a+r}{\sqrt{2R}}\overset{\text{Bogdan Fustei}}{=}$$

$$2.\sqrt{\frac{h_a\left(n_a-\frac{an_a}{s}\right)}{n_a}}+\frac{r_a+r}{\sqrt{2R}}=2.\sqrt{\frac{2rs(s-a)}{as}}+\frac{\frac{rs}{s-a}+\frac{rs}{s}}{\sqrt{2R}}$$

$$=2.\sqrt{\frac{2.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}}+\frac{rs(b+c)}{\sqrt{2R}.s(s-a)}$$

$$=4.\sqrt{2R}.\sin\frac{B}{2}\sin\frac{C}{2}+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sqrt{2R}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=2.\sqrt{2R}.\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)+2.\sqrt{2R}.\sin\frac{A}{2}\cos\frac{B-C}{2}$$

$$\le 2.\sqrt{2R}.\left(1-\sin\frac{A}{2}+\sin\frac{A}{2}\right)\ \left(\because -\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}\Rightarrow 0<\cos\frac{B-C}{2}\le 1\right)=2.\sqrt{2R}$$

$$\therefore 2.\sqrt{\frac{h_a\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}{n_a}}+\frac{r_a+r}{\sqrt{2R}}\le 2\sqrt{2R}\ \text{and also,}$$

$$\sqrt{\frac{h_a}{n_a}.2R\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\overset{\text{Bogdan Fustei}}{=}\sqrt{\frac{h_a}{n_a}.2R\left(n_a-\frac{an_a}{s}\right)}$$

$$=\sqrt{\frac{2rs}{sn_a}.2Rn_a.\frac{s-a}{a}}=2\sqrt{R}.\sqrt{\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}}$$

$$=2R\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\overset{\cos\frac{B-C}{2}\le 1}{\le}2R\left(1-\sin\frac{A}{2}\right)$$

$$\therefore\sqrt{\frac{h_a}{n_a}.2R\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\le 2R\left(1-\sin\frac{A}{2}\right)\to (1)$$

$$\text{Again}, \frac{r_b + r_c}{2} - r = 2R\cos^2\frac{A}{2} - 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)$$

$$= 2R\left(\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - \sin\frac{A}{2}\cos\frac{B-C}{2}\right) \overset{\cos\frac{B-C}{2} \le 1}{\ge} 2R\left(1 - \sin\frac{A}{2}\right) \overset{\text{via } (1)}{\ge}$$

$$\sqrt{\frac{h_a}{n_a}.2R\left(n_a - \sqrt{4r^2 + (b-c)^2}\right)}$$

$$\therefore \sqrt{2R\left(n_a - \sqrt{4r^2 + (b-c)^2}\right)} \le \left(\frac{r_b + r_c}{2} - r\right).\sqrt{\frac{n_a}{h_a}} \text{ and hence,}$$

both inequalities hold true $\forall \Delta ABC$, " $=$ " iff $b = c$ (QED)