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If in $\Delta ABC, I-$ incenter, then:

$$\sqrt{\frac{R}{2r}}\cdot\sum_{cyc}(g_a+|b-c|)\geq\sum_{cyc}\left(AI.\frac{n_a}{h_a}\right)$$

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We will first prove that

$$g_a+|b-c|\geq n_a.\quad(1)$$

We have $n_a{}^2=s(s-a)+\dfrac{s(b-c)^2}{a}$ **and** $g_a{}^2=s(s-a)-\dfrac{(s-a)(b-c)^2}{a}$, **then**

$$(n_a g_a)^2=\big(s(s-a)\big)^2+s(s-a)\left(1-\frac{(b-c)^2}{a^2}\right)(b-c)^2\overset{b>|c-a|}{\geq}\big(s(s-a)\big)^2\Rightarrow n_a g_a\geq s(s-a)$$

$$\Rightarrow\ (n_a-g_a)^2=\left(n_a{}^2+g_a{}^2\right)-2n_a g_a\leq\left(2s(s-a)+(b-c)^2\right)-2s(s-a)=(b-c)^2,$$

then the proof of (1) is complete. Also, we have:

$$\frac{AI}{h_a}=\frac{r}{\sin\frac{A}{2}}\cdot\frac{a}{2F}=\frac{2R}{s}\cos\frac{A}{2}=\frac{2R}{s}\sqrt{\frac{s(s-a)}{bc}}=2R\sqrt{\frac{a(s-a)}{s.4Rsr}}=\sqrt{\frac{R}{2r}}\cdot\sqrt{\frac{2a(s-a)}{s^2}}\leq\sqrt{\frac{R}{2r}},$$

$\left(\because s^2-2a(s-a)=(s-a)^2+a^2\right).$

Then :

$$\sqrt{\frac{R}{2r}}.(g_a+|b-c|)\geq\frac{AI}{h_a}.n_a=AI.\frac{n_a}{h_a}\quad\text{(and analogs)}$$

Therefore :

$$\sqrt{\frac{R}{2r}}\cdot\sum_{cyc}(g_a+|b-c|)\geq\sum_{cyc}\left(AI.\frac{n_a}{h_a}\right)$$