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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{n_a}{n_a-2r}\le 1+\frac{4R}{r}\ \ and\ \sum_{cyc}\frac{n_a}{n_a-2r}\le\sum_{cyc}\frac{h_a}{h_a-2r}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{n_a}{n_a-2r}=\sum_{cyc}\frac{1}{1-\frac{2r}{n_a}}\le\sum_{cyc}\frac{1}{1-\frac{2r}{h_a}}=\sum_{cyc}\frac{1}{1-\frac{a}{s}}$$

$$=\frac{s}{r^2s}\cdot\sum_{cyc}\big((s-b)(s-c)\big)=\frac{4Rr+r^2}{r^2}\ \therefore\ \sum_{cyc}\frac{n_a}{n_a-2r}\le 1+\frac{4R}{r}\ \ and$$

$$\because\ n_a\ge h_a\ and\ analogs\ \therefore\ \frac{n_a}{n_a-2r}\le\frac{h_a}{h_a-2r}\ and\ analogs$$

$$\Rightarrow\sum_{cyc}\frac{n_a}{n_a-2r}\le\sum_{cyc}\frac{h_a}{h_a-2r}\ \therefore\ \sum_{cyc}\frac{n_a}{n_a-2r}\le 1+\frac{4R}{r}\ \ and$$

$$\sum_{cyc}\frac{n_a}{n_a-2r}\le\sum_{cyc}\frac{h_a}{h_a-2r},\ ''=''\ \text{ iff }\ \Delta\,ABC\text{ is equilateral (QED)}$$