ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $following$ $relationship$ $holds$:

$$|b-c| \le 2\sqrt{R(R-2r)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We have : } R-2r \overset{?}{\ge} \frac{b^2+c^2}{4R} - \frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right) \overset{?}{\ge} \frac{4R^2(\sin^2B+\sin^2C)}{4R} - \frac{4R^2\sin B\sin C}{2R}$$

$$\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} \overset{?}{\ge} \sin^2B+\sin^2C-2\sin B\sin C = (\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right) \overset{?}{\ge} \left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right) \overset{?}{\ge} 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2} \overset{?}{\ge} 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1 \overset{?}{\ge} 0 \Leftrightarrow \left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2 \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \Rightarrow 4R(R-2r) \ge (b-c)^2 \Rightarrow |b-c| \le 2\sqrt{R(R-2r)} \ \ \forall\, \Delta\, ABC,$$

$$'' ='' \ \text{ iff } \ \cos\frac{B-C}{2} = \frac{1}{2\sin\frac{A}{2}}, \text{with } \Delta\, ABC \text{ is } equilateral \text{ being } a \text{ } special \text{ case (QED)}$$