ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\frac{R}{r}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)\geq\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}$$

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$$n_a^2\overset{\text{Bogdan Fustei}}{=}s^2-2r_ah_a\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)$$

$$\overset{?}{\leq}4(R^2-2Rr+r^2)s^2\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0$$

$$\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to\text{true}\therefore an_a\leq 2Rs-2rs$$

$$\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}\Rightarrow\frac{n_a}{h_a}\leq\frac{R}{r}-1$$

$$\Rightarrow\left(\frac{R}{r}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)\geq\frac{n_a\left(n_a-s+\sqrt{2r_ah_a}\right)}{h_a\left(s-\sqrt{2r_ah_a}\right)}$$

$$\left(\begin{array}{c}\because n_a-s+\sqrt{2r_ah_a}\overset{\text{Fustei}}{=}\sqrt{s^2-n_a^2}-(s-n_a)=\sqrt{s-n_a}.\left(\sqrt{s+n_a}-\sqrt{s-n_a}\right)\\ \overset{\text{Fustei}}{=}\sqrt{\frac{2r_ah_a}{s+n_a}}.\frac{2n_a}{\sqrt{s+n_a}+\sqrt{\frac{2r_ah_a}{s+n_a}}}>0\end{array}\right)$$

$$=\frac{n_a\left(n_a-s+\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)}{h_a(s^2-2r_ah_a)}\overset{\text{Bogdan Fustei}}{=}\frac{n_a\left(n_a\left(s+\sqrt{2r_ah_a}\right)-(s^2-2r_ah_a)\right)}{h_an_a^2}$$

$$\overset{\text{Fustei}}{=}\frac{n_a\left(n_a\left(s+\sqrt{2r_ah_a}\right)-n_a^2\right)}{h_an_a^2}=\frac{s+\sqrt{2r_ah_a}-n_a}{h_a}=\frac{s-n_a}{h_a}+\sqrt{\frac{2r_a}{h_a}}$$

$$=\frac{s^2-n_a^2}{h_a(s+n_a)}+\sqrt{\frac{2r_a}{h_a}}\overset{\text{Bogdan Fustei}}{=}\frac{2r_ah_a}{h_a(s+n_a)}+\sqrt{\frac{2r_a}{h_a}}=\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}$$

$$\therefore\left(\frac{R}{r}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)\geq\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}\ \forall\Delta\,ABC,''='' \text{ iff } b=c\ (QED)$$