ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC the following relationship holds :

$$\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}\ge\left(\frac{b}{c}+\frac{c}{b}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)$$

*Proposed by Bogdan Fuștei-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\left(\frac{b}{c}+\frac{c}{b}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)=$$

$$=\frac{b^2-bc+c^2}{bc}\cdot\frac{\left(n_a-s+\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)}{s^2-2r_ah_a}=$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{b^2-bc+c^2}{bc}\cdot\frac{\left(n_a-s+\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)}{n_a^2}\overset{\text{Ben Ajiba}}{\le}$$

$$\le\frac{b^2-bc+c^2}{bc}\cdot\frac{\left(n_a-s+\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)}{n_a\cdot\frac{b^2-bc+c^2}{2R}}$$

$$\left(\begin{array}{c}\because n_a-s+\sqrt{2r_ah_a}\overset{\text{Fustei}}{=}\sqrt{s^2-n_a^2}-(s-n_a)=\sqrt{s-n_a}\cdot\left(\sqrt{s+n_a}-\sqrt{s-n_a}\right)\\ \overset{\text{Fustei}}{=}\sqrt{\frac{2r_ah_a}{s+n_a}}\cdot\frac{2n_a}{\sqrt{s+n_a}+\sqrt{\frac{2r_ah_a}{s+n_a}}}>0\end{array}\right)$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba}-28;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$=\frac{\left(n_a-s+\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)}{n_ah_a}=\frac{n_a\left(s+\sqrt{2r_ah_a}\right)-\left(s^2-2r_ah_a\right)}{n_ah_a}=$$

$$\overset{\text{Fustei}}{=}\frac{n_a\left(s+\sqrt{2r_ah_a}\right)-n_a^2}{n_ah_a}=\frac{s+\sqrt{2r_ah_a}-n_a}{h_a}=\frac{s-n_a}{h_a}+\sqrt{\frac{2r_a}{h_a}}=$$

$$=\frac{s^2-n_a^2}{h_a(s+n_a)}+\sqrt{\frac{2r_a}{h_a}}\overset{\text{Bogdan Fustei}}{=}\frac{2r_ah_a}{h_a(s+n_a)}+\sqrt{\frac{2r_a}{h_a}}=\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}$$

$$\therefore\frac{2r_a}{s+n_a}+\sqrt{\frac{2r_a}{h_a}}\ge\left(\frac{b}{c}+\frac{c}{b}-1\right)\left(\frac{n_a}{s-\sqrt{2r_ah_a}}-1\right)\ \forall\,\Delta\,ABC,\text{''}=\text{'' iff } b=c\ (\text{QED})$$