In any Δ ABC the following relationship holds :

$$\frac{n_a.|r_b - r_c|}{p_a.\sqrt{\frac{w_a}{g_a}} - \sqrt{4r^2 + (b-c)^2}} \geq \frac{s|b-c|}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2 + \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} + 2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) + \left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} + \frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} + \frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a w_a \rightarrow (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2.m_a w_a$$

$$\Rightarrow \frac{n_a}{p_a} \geq \sqrt{\frac{w_a}{g_a}} \Rightarrow p_a.\sqrt{\frac{w_a}{g_a}} - \sqrt{4r^2 + (b-c)^2} \leq n_a - \sqrt{4r^2 + (b-c)^2} \overset{\text{Bogdan Fustei}}{=}$$

$$\mathbf{n}_a - \frac{a\mathbf{n}_a}{\mathbf{s}} \Rightarrow \frac{\mathbf{n}_a.\,|\mathbf{r}\mathbf{r}_\mathrm{b} - \mathbf{r}\mathbf{r}_\mathrm{c}|}{\mathbf{p}_a.\sqrt{\frac{\mathrm{w}_a}{g_a}} - \sqrt{4\mathbf{r}^2 + (\mathbf{b} - \mathbf{c})^2}} \geq \frac{\mathbf{s}.\,\mathbf{n}_a.\,|(\mathbf{s} - \mathbf{c})(\mathbf{s} - a) - (\mathbf{s} - a)(\mathbf{s} - \mathbf{b})|}{\mathbf{n}_a.\,(\mathbf{s} - a)}$$

$$= \mathbf{s}|\mathbf{b} - \mathbf{c}| \Rightarrow \frac{\mathbf{n}_a.\,|\mathbf{r}_\mathrm{b} - \mathbf{r}_\mathrm{c}|}{\mathbf{p}_a.\sqrt{\frac{\mathrm{w}_a}{g_a}} - \sqrt{4\mathbf{r}^2 + (\mathbf{b} - \mathbf{c})^2}} \geq \frac{\mathbf{s}|\mathbf{b} - \mathbf{c}|}{\mathbf{r}}\ \forall\,\Delta\,\mathbf{ABC},$$

$'' ='' $ **iff Δ ABC is equilateral (QED)**