ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{2n_a\left(3m_a - n_a - \sqrt{r_b r_c}\right)}{h_a} \geq \frac{n_a g_a}{h_a} + \frac{r_b + r_c}{2}$$

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$$n_a + g_a \overset{\text{CBS}}{\leq} \sqrt{2(n_a^2 + g_a^2)} \overset{\text{Bogdan Fustei}}{=} \sqrt{2\left(4m_a^2 - 2s(s-a)\right)}$$

$$= 2.\sqrt{2m_a^2 - s(s-a)} \overset{?}{\leq} 4m_a - 2\sqrt{s(s-a)}$$

$$\Leftrightarrow \left(2m_a - \sqrt{s(s-a)}\right)^2 \overset{?}{\geq} 2m_a^2 - s(s-a)$$

$$\Leftrightarrow 4m_a^2 + s(s-a) - 4m_a.\sqrt{s(s-a)} \overset{?}{\geq} 2m_a^2 - s(s-a)$$

$$\Leftrightarrow m_a^2 + s(s-a) - 2m_a.\sqrt{s(s-a)} \overset{?}{\geq} 0 \Leftrightarrow \left(m_a - \sqrt{s(s-a)}\right)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore n_a + g_a \leq 4m_a - 2\sqrt{s(s-a)} \Rightarrow 4m_a - n_a - g_a - 2\sqrt{r_b r_c} \geq 0 \textit{ and so,}$$

$$\text{in order to prove : } 6m_a \geq g_a + 2n_a + 2\sqrt{r_b r_c} + \frac{r_b r_c}{n_a}, \text{it suffices to prove :}$$

$$2m_a \geq n_a + \frac{r_b r_c}{n_a} \to \text{true via Bogdan Fustei}$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Bogdan Fustei} - 42;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\therefore 6m_a \geq g_a + 2n_a + 2\sqrt{r_b r_c} + \frac{r_b r_c}{n_a} \Rightarrow \frac{6m_a n_a - 2n_a^2 - 2n_a.\sqrt{r_b r_c}}{h_a} \geq \frac{n_a g_a}{h_a} + \frac{r_b r_c}{h_a}$$

$$= \frac{n_a g_a}{h_a} + \frac{bc.\cos^2\frac{A}{2}}{\frac{2rs}{a}} = \frac{n_a g_a}{h_a} + \frac{4Rrs.\cos^2\frac{A}{2}}{2rs} = \frac{n_a g_a}{h_a} + \frac{4R\cos^2\frac{A}{2}}{2} = \frac{n_a g_a}{h_a} + \frac{r_b + r_c}{2}$$

$$\therefore \frac{2n_a\left(3m_a - n_a - \sqrt{r_b r_c}\right)}{h_a} \geq \frac{n_a g_a}{h_a} + \frac{r_b + r_c}{2}\ \forall\, \Delta\, ABC, ''='' \text{ iff } b = c \text{ (QED)}$$