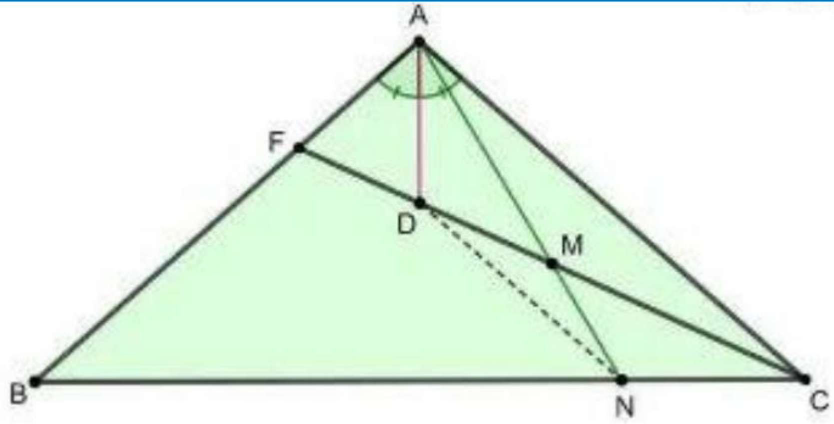
ROMANIAN MATHEMATICAL MAGAZINE



$$\Delta ABC, F \in AB, FM = MC, AM \cap BC = N, < BAD =< DAC,$$

$$DN \,||\, AC \;\; , AF = AB/k$$

$$Prove: DN = \frac{k-1}{k+1} AB$$

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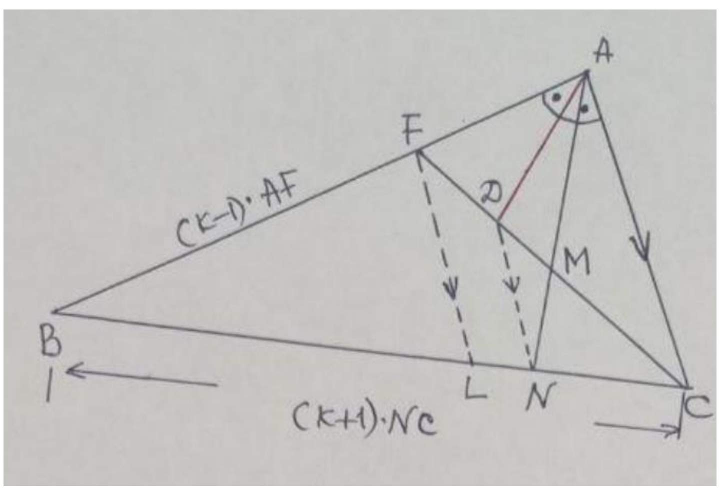
Solution by Mirsadix Muzefferov-Azerbaijan



$$Let's\ draw\ the\ line\ FL\,||\,DN. Then\ we\ get\ similar\ triangles:$$

$$\Delta LFC \sim \Delta NDC => \frac{DN}{FL} = \frac{DC}{FC}$$

$$\Delta DNM \sim \Delta AMC => \frac{DN}{AC} = \frac{DM}{MC} => DN = \frac{DM}{MC} AC \text{ (*)}$$

$$\Delta BFL \sim \Delta BAC => \frac{BF}{AB} = \frac{FL}{AC} => AC = \frac{FL}{BF} AB \ (**)$$

$$In\ \Delta AFC\ AD\ is\ bisector => \frac{AF}{FD} = \frac{AC}{DC} => \frac{DC}{FD} = \frac{AC}{AF}\ (1)$$

$$(*) => DN = \frac{DM}{MC} AC =^{(**)} \frac{DM}{MC} \times \frac{FL}{BF} \times AB = \frac{DM}{MC} \times \frac{FL}{(k-1)AF} AB$$

$$\frac{BF}{AB} = \frac{FL}{AC} =^{(1)} \frac{FL}{\frac{AF \times DC}{FD}} = \frac{FL \times FD}{AF \times DC}; => \frac{FL}{AF} = \frac{BF \times DC}{AB \times FD}\ (2)$$

$$DN = \frac{DM}{MC} \times \frac{FL}{(k-1)AF} \times AB = \frac{DM}{\frac{FC}{2}} \times \frac{FL}{(k-1)AF} \times AB = \frac{2DM}{FC} \times \frac{FL}{(k-1)AF} \times AB =$$

$$=^{(2)} \frac{2DM}{FC} \times \frac{1}{k-1} \times \frac{BF \times DC}{AB \times FD} \times AB = \frac{2DM}{FC} \times \frac{1}{k-1} \times \frac{(k-1)AF \times DC}{k \times AF \times FD} \times AB =$$

$$= \frac{2DM \times DC}{FC \times k \times FD} \times AB = \frac{2(DC - MC) \times DC}{FC \times k(FC - DC)} = \frac{2\left(DC - \frac{FC}{2}\right) DC}{FC \times k(FC - DC)} =$$

$$= \frac{(2DC - FC)DC}{k \times FC(FC - DC)} \times AB = \frac{(2DC - DF - DC)DC}{k \times (FD + DC)(FD + DC - DC)} \times AB =$$

$$\frac{(DC - FD)DC}{k \times FD(FD + DC)} \times AB = \frac{1}{k} \times \frac{DC}{FD} \times \frac{DC - FD}{DC + FD} \times AB = \frac{1}{k} \times \frac{DC}{DF} \times \frac{\frac{DC}{FD} - 1}{\frac{DC}{FD} + 1} \times AB$$

$$Let's\ prove\ that \frac{DC}{FD} = k.$$

$$For\ this, let\ us\ consider\ the\ Menelaus\ theorem\ in\ \Delta ABC$$

$$\frac{NC}{BC} \times \frac{BF}{FA} \times \frac{AM}{MN} = 1, \frac{AF}{AB} \times \frac{BN}{NC} \times \frac{CM}{MF} = 1$$

$$\frac{1}{k+1} \times \frac{k-1}{1} \times \frac{AM}{MN} = 1, \frac{1}{k} \times \frac{BN}{NC} = 1 => \frac{BN}{NC} = k.$$

$$\frac{MN}{AM} = \frac{k-1}{k+1},$$

$$\Delta DMN \infty \Delta AMC => \frac{MN}{AM} = \frac{DN}{AC} => \frac{DN}{AC} = \frac{k-1}{k+1}$$

$$\frac{DM}{MC}=\frac{DN}{AC}=\frac{k-1}{k+1};$$

$$\frac{FD}{MC}=\frac{FC-DC}{MC}=\frac{FC-DM-MC}{MC}=2-\frac{DM}{MC}-1=1-\frac{DM}{MC}=$$

$$=1-\frac{k-1}{k+1}=\frac{2}{k+1}$$

$$\frac{DC}{FD}=\frac{(DM+MC)}{FD}=\frac{\left(\frac{DM}{MC}+1\right)}{\frac{FD}{MC}}=\frac{\frac{k-1}{k+1}+1}{\frac{2}{k+1}}=\frac{\frac{2k}{k+1}}{\frac{2}{k+1}}=k$$

$$So, DN=\frac{1}{k}\times k\times\frac{k-1}{k+1}\times AB=\frac{k-1}{k+1}\times AB\,.$$