ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\sum\frac{1}{a^2+b^2}\left(b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\right)\ge\frac{9\sqrt{2}(abc)^{\frac{5}{3}}}{a^2+b^2+c^2}$$

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Solution by Tapas Das-India

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\overset{Cesaro}{\ge}8a^2b^2c^2\ (1)$$

$$\sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\overset{AM-GM}{\le}\frac{2(a^2+b^2+c^2)}{3}\ (2)$$

$$b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\overset{AM-GM}{\ge}2\sqrt{b^4c^4}.\sqrt[4]{(b^2+c^2)(c^2+a^2)}=$$

$$=2b^2c^2\sqrt[4]{(b^2+c^2)(c^2+a^2)}(3)$$

$$\sum\frac{1}{a^2+b^2}\left(b^4\sqrt{b^2+c^2}+c^4\sqrt{c^2+a^2}\right)\overset{(2),(3)}{\ge}\sum\frac{2b^2c^2.\sqrt[4]{(b^2+c^2)(c^2+a^2)}}{a^2+b^2}\overset{AM-GM}{\ge}$$

$$\ge 6\sqrt[3]{\frac{a^4b^4c^4\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\overset{(1)}{\ge}$$

$$\ge 6\sqrt[3]{\frac{a^4b^4c^42^{\frac{3}{2}}\,abc}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\overset{(2)}{\ge}\frac{6\sqrt{2}(abc)^{\frac{5}{3}}}{\frac{2(a^2+b^2+c^2)}{3}}=\frac{9\sqrt{2}(abc)^{\frac{5}{3}}}{a^2+b^2+c^2}$$

Equality holds for $a=b=c$.