ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ *then prove that* :

$$\sum_{cyc} \frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4} \ge \frac{6}{\sqrt[3]{abc}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B') \ge \sqrt{3\sum_{cyc} A'B'}$$

(via Walter Janous) $\rightarrow$ ① and now, $\displaystyle\sum_{cyc} \frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4} =$

$$\sum_{cyc} \frac{\frac{1}{a^4}\left(\frac{b^3+c^3}{b^4}+\frac{c^3+a^3}{c^4}\right)}{\frac{1}{b^4}+\frac{1}{c^4}} = \frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x' = \frac{1}{a^4}, y' = \frac{1}{b^4}, z' = \frac{1}{c^4}, A' = \frac{a^3+b^3}{a^4}, B' = \frac{b^3+c^3}{b^4}, C' = \frac{c^3+a^3}{c^4}\right) \overset{\text{via } ①}{\ge}$$

$$\sqrt{3\sum_{cyc}\left(\frac{a^3+b^3}{a^4}\cdot\frac{b^3+c^3}{b^4}\right)} \overset{\substack{\text{AM-GM} \\ \text{and} \\ \text{Cesaro}}}{\ge} 3.\sqrt[6]{\left(\frac{8a^3b^3c^3}{a^4b^4c^4}\right)^2} = \frac{6}{\sqrt[3]{abc}}\ \forall\, a, b, c > 0,$$

$$'' ='' \text{ iff } a = b = c\ )$$

Solution 2 by Mirsadix Muzefferov-Azerbaijan

Let's transform the left side of the expression according to the Walter Janous inequality

$$\frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4} = \frac{\frac{b^4c^3}{a^4}+\frac{b^4}{a}+\frac{c^4b^3}{a^4}+\frac{c^7}{a^4}}{b^4+c^4} =$$

$$= \frac{\frac{1}{a^4}(b^4c^3+a^3b^4+c^4b^3+c^7)}{b^4+c^4} = \frac{\frac{1}{a^4}\cdot\frac{b^4c^3+a^3b^4+c^4b^3+c^7}{b^4.c^4}}{\frac{b^4+c^4}{b^4.c^4}} =$$

$$= \frac{\frac{1}{a^4}\left(\frac{a^3+c^3}{c^4}+\frac{b^3+c^3}{b^4}\right)}{\frac{1}{b^4}+\frac{1}{c^4}} = \frac{\frac{1}{a^4}}{\frac{1}{b^4}+\frac{1}{c^4}}\left(\frac{a^3+c^3}{c^4}+\frac{b^3+c^3}{b^4}\right) = \frac{x}{y+z}(B+C)$$

Walter Janous inequality $x, y, z, A, B, C > 0$

$$(*)\ \frac{x}{y+z}(B+C)+\frac{y}{x+z}(A+C)+\frac{z}{y+x}(B+A)\geq\sqrt{3(AB+BC+CA)}$$

$$\textit{Analogously}\ :\ \frac{\frac{1}{b^4}}{\frac{1}{a^4}+\frac{1}{c^4}}\left(\frac{a^3+b^3}{a^4}+\frac{a^3+c^3}{c^4}\right)=\frac{y}{x+z}(A+C)$$

$$\frac{\frac{1}{c^4}}{\frac{1}{a^4}+\frac{1}{b^4}}\left(\frac{c^3+b^3}{b^4}+\frac{a^3+b^3}{a^4}\right)=\frac{z}{x+y}(A+B)$$

$$\textit{Here}:\left(x=\frac{1}{a^4},y=\frac{1}{b^4},z=\frac{1}{c^4},A=\frac{a^3+b^3}{a^4},B=\frac{c^3+b^3}{b^4},C=\frac{a^3+c^3}{c^4}\right)$$

$$LHS\overset{\textit{Walter Janous}}{\geq}\sqrt{3(AB+BC+CA)}\overset{AM-GM}{\geq}\sqrt{3.3\sqrt[3]{A^2.B^2.C^2}}=$$

$$=3.\sqrt[3]{ABC}=3\sqrt[3]{\frac{a^3+b^3}{a^4}.\frac{c^3+b^3}{b^4}.\frac{a^3+c^3}{c^4}}\overset{AM-GM}{\geq}$$

$$\geq3\sqrt[3]{\frac{2\sqrt{a^3b^3}}{a^4}.\frac{2\sqrt{b^3c^3}}{b^4}.\frac{2\sqrt{c^3a^3}}{c^4}}=6\sqrt[3]{\frac{a^3b^3c^3}{a^4b^4c^4}}=\frac{6}{\sqrt[3]{abc}}$$