ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ then prove that :

$$\frac{(3a^2+2b^2+2ca+3bc)(3b^2+2c^2+2ab+3ca)(3c^2+2a^2+2bc+3ab)}{(2b+3c)(2c+3a)(2a+3b).\sqrt{(2b+3c)(2c+3a)(2a+3b)abc}} \geq \frac{8\sqrt{5}}{25}$$

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$$\mathrm{LHS} = \frac{\left(\sqrt{\begin{pmatrix} b(2b+3c) \\ +a(2c+3a)\end{pmatrix}\begin{pmatrix} c(2c+3a) \\ +b(2a+3b)\end{pmatrix}\begin{pmatrix} a(2a+3b) \\ +c(2b+3c)\end{pmatrix}}\right).\prod_{cyc}\sqrt{\frac{b(2b+3c)+a(2c+3a)}{(2b+3c)(2c+3a)}}}{\sqrt{(2b+3c)(2c+3a)(2a+3b)abc}}$$

$$\overset{AM-GM}{\geq} \frac{\sqrt{8\sqrt{(abc)^2(2b+3c)^2(2c+3a)^2(2a+3b)^2}}}{\sqrt{(2b+3c)(2c+3a)(2a+3b)abc}}.\prod_{cyc}\sqrt{\frac{b}{2c+3a}+\frac{a}{2b+3c}}$$

$$= 2\sqrt{2}.\sqrt{\left(\frac{a}{2b+3c}+\frac{b}{2c+3a}\right)\left(\frac{b}{2c+3a}+\frac{c}{2a+3b}\right)\left(\frac{c}{2a+3b}+\frac{a}{2b+3c}\right)}$$

$$= 2\sqrt{2}.\sqrt{\left(\sum_{cyc}\frac{a}{2b+3c}\right)\left(\sum_{cyc}\frac{ab}{(2b+3c)(2c+3a)}\right) - \frac{abc}{(2a+3b)(2b+3c)(2c+3a)}}$$

$$\overset{\substack{\text{Bergstrom} \\ \text{and} \\ AM-GM}}{\geq} = 2\sqrt{2}.\sqrt{\left(\frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab+3\sum_{cyc}ca}\right)\left(\sum_{cyc}\frac{ab(2a+3b)}{(2a+3b)(2b+3c)(2c+3a)}\right) - \frac{abc}{(5.\sqrt[5]{a^2b^3})(5.\sqrt[5]{b^2c^3})(5.\sqrt[5]{c^2a^3})}}$$

$$\geq 2\sqrt{2}.\sqrt{\left(\frac{3\sum_{cyc}ab}{5\sum_{cyc}ab}\right)\left(\sum_{cyc}\frac{ab(2a+3b)}{(2a+3b)(2b+3c)(2c+3a)}\right) - \frac{1}{125}} \overset{?}{\geq} \frac{8.\sqrt{5}}{25}$$

$$\Leftrightarrow \sum_{cyc}ab(2a+3b) \overset{?}{\geq} \frac{3}{25}.(2a+3b)(2b+3c)(2c+3a)$$

$$\Leftrightarrow 2\sum_{cyc}a^2b + 3\sum_{cyc}ab^2 \overset{?}{\geq} 15abc \to \text{true} \because \sum_{cyc}a^2b, \sum_{cyc}ab^2 \overset{AM-GM}{\geq} 3abc$$

$$\therefore \frac{(3a^2+2b^2+2ca+3bc)(3b^2+2c^2+2ab+3ca)(3c^2+2a^2+2bc+3ab)}{(2b+3c)(2c+3a)(2a+3b).\sqrt{(2b+3c)(2c+3a)(2a+3b)abc}}$$

$$\geq \frac{8.\sqrt{5}}{25}\ \forall a,b,c>0, ''='' \text{ iff } a=b=c\ (QED)$$