ROMANIAN MATHEMATICAL MAGAZINE

If $x, y \in (0,1)$ then prove that :

$$\sqrt{x}+\sqrt{y}+\sqrt[4]{12}\cdot\sqrt{y\sqrt{1-x^2}+x\sqrt{1-y^2}}\le 4\sqrt[4]{\frac{2}{3}}$$

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$$\sqrt{x}+\sqrt{y}+\sqrt[4]{12}.\sqrt{y.\sqrt{1-x^2}+x.\sqrt{1-y^2}}\overset{\text{CBS}}{\le}$$

$$\sqrt{2}.\sqrt{\sqrt{2(x^2+y^2)}}+\sqrt[4]{12}.\sqrt{\sqrt{x^2+y^2}.\sqrt{2-(x^2+y^2)}}$$

$$=\sqrt[4]{8(x^2+y^2)}+\sqrt[4]{24(x^2+y^2)-12(x^2+y^2)^2}$$

$$=t+\sqrt[4]{24.\frac{t^4}{8}-12.\left(\frac{t^4}{8}\right)^2}\ \left(t=\sqrt[4]{8(x^2+y^2)}\right)=t+\frac{t}{2}.\sqrt[4]{48-3t^4}$$

$$\therefore \text{LHS}\overset{(1)}{\le} t+\frac{t}{2}.\sqrt[4]{48-3t^4}\ \textit{and since}\ x,y\in(0,1)\ \therefore t=\sqrt[4]{8(x^2+y^2)}\in(0,2)$$

$$\text{Let } f(t)=t+\frac{t}{2}.\sqrt[4]{48-3t^4}\ \forall\, t\in(0,2);\text{then : } f'(t)=\frac{(48-3t^4)^{\frac{3}{4}}-3(t^4-8)}{(48-3t^4)^{\frac{3}{4}}}>0$$

$$(\text{trivi}\mathit{ally})\text{ if } t^4\le 8\text{nd } \forall\, t^4\in(8,16), f'(t)=$$

$$\frac{(48-3t^4)^3-81(t^4-8)^4}{(48-3t^4)^{\frac{3}{4}}.\left((48-3t^4)^{\frac{3}{4}}+3(t^4-8)\right).\left((48-3t^4)^{\frac{3}{2}}+9(t^4-8)^2\right)}$$

$$=\frac{27(32-3m)\left((m-5)(m-8)^2+64\right)}{(48-3t^4)^{\frac{3}{4}}.\left((48-3t^4)^{\frac{3}{4}}+3(t^4-8)\right).\left((48-3t^4)^{\frac{3}{2}}+9(t^4-8)^2\right)}\overset{(*)}{=}f'(t)$$

$$(m=t^4)\ \textit{and}\ \because m=t^4>8\ \therefore f'(t)=0\Rightarrow 32=3m=3t^4\Rightarrow t=\sqrt[4]{\frac{32}{3}}\ \textit{and also,}$$

$$(*)\Rightarrow \forall\, m=t^4\in\left(0,\frac{32}{3}\right], f'(t)\ge 0\ \textit{and}\ \forall\, m=t^4\in\left(\frac{32}{3},16\right), f'(t)<0$$

$$\therefore f(t)\text{ is}\uparrow\text{on}\left(0,\sqrt[4]{\frac{32}{3}}\right]\ \textit{and}\ f(t)\text{ is}\downarrow\text{on}\left(\sqrt[4]{\frac{32}{3}},2\right)\text{ with } f'\left(\sqrt[4]{\frac{32}{3}}\right)=0$$

$$\therefore\ \mathbf{f(t)_{max} = f\left(\sqrt[4]{\frac{32}{3}}\right) = 4.\sqrt[4]{\frac{2}{3}} \overset{via\ ①}{\Rightarrow} \sqrt{x} + \sqrt{y} + \sqrt[4]{12}.\sqrt{y.\sqrt{1-x^2} + x.\sqrt{1-y^2}}}$$

$$\mathbf{\leq 4.\sqrt[4]{\frac{2}{3}},\ ''='' \ \text{iff}\ x = y = \sqrt{\frac{2}{3}}\ (QED)}$$