ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ then:

$$\frac{1}{\sqrt{a^2-ab+3b^2+1}}+\frac{1}{\sqrt{b^2-bc+3c^2+1}}+\frac{1}{\sqrt{c^2-ca+3a^2+1}}\le\frac{3}{2}$$

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Solution by Tapas Das-India

$$a^2-ab+3b^2+1=(a^2+b^2)+(b^2+1)+b^2-ab\overset{AM-GM}{\ge}$$

$$\ge 2ab+2b+b^2-ab=b(a+b+2)\ (1)$$

$$\frac{1}{\sqrt{a^2-ab+3b^2+1}}+\frac{1}{\sqrt{b^2-bc+3c^2+1}}+\frac{1}{\sqrt{c^2-ca+3a^2+1}}=$$

$$=\sum\frac{1}{\sqrt{a^2-ab+3b^2+1}}\overset{(1)}{\le}\sum\frac{1}{\sqrt{b(a+b+2)}}=$$

$$=2\sum\sqrt{\frac{1}{4b}\cdot\frac{1}{a+b+2}}\overset{AM-GM}{\le}\sum\left(\frac{1}{4b}+\frac{1}{a+b+2}\right)=$$

$$=\sum\frac{1}{4b}+\sum\frac{1}{a+b+1+1}\overset{AM-HM}{\le}\frac{1}{4}\sum\frac{1}{b}+\frac{1}{16}\sum\left(\frac{1}{a}+\frac{1}{b}+1+1\right)=$$

$$=\frac{1}{4}\sum\frac{1}{b}+\frac{1}{16}\left(\sum\frac{1}{a}+\sum\frac{1}{b}+\sum 1+\sum 1\right)\overset{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3}{=}$$

$$=\frac{3}{4}+\frac{1}{16}(3+3+3+3)=\frac{3}{2}$$

Equality holds for a=b=c=1.