ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2}\le x^2+y^2+32$$

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$\boldsymbol{Lemma}$:

$$\forall\, a,b>0\,,\sqrt{22a^2+36ab+6b^2}\le(5a+3b)$$

$\boldsymbol{Proof}$:

$$\boldsymbol{We\ need\ to\ show}:$$

$$\sqrt{22a^2+36ab+6b^2}\le(5a+3b)$$

$$22a^2+36ab+6b^2\overset{squaring}{\le}25a^2+30ab+9b^2$$

$$3(a^2-2ab+b^2)\ge 0 \text{ or } 3(a-b)^2\ge 0 \text{ true.}$$

$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2}\overset{Lemma}{\le}$$
$$\le(5x+3y)+(5y+3x)=8(x+y)\ (1)$$

$$\boldsymbol{We\ need\ to\ show}:$$

$$\sqrt{22x^2+36xy+6y^2}+\sqrt{22y^2+36xy+6x^2}\le x^2+y^2+32$$

$$8(x+y)\overset{(1)}{\le}x^2+y^2+32 \text{ or } \frac{(x+y)^2}{2}+32\overset{CBS}{\ge}8(x+y)$$

$$(x+y)^2-16(x+y)+64\ge 0 \text{ or } (x+y-8)^2\ge 0$$

$$\text{Equality holds for } x=y=4$$